Model of categories for image processing

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Abstract

Theory of categories has various applications in technology. There are mathematical models that have been built using algebraic structures equipped with different tools of properties from disciplines inside mathematics even inside algebra. We are concentrated on the applications of algebra in image processing. We have seen that basic relations and operations between images, and other used in image compression, already have been described in terms of categories and homomorphism of modules over quantales. They include relations and operations between sets, set inclusion, union, intersection, used in mathematical morphology, and operations, order relation, t-norms in image compression. Focused on the way how these construction of categories of quantale modules are built up we want to show a new construction of an algebraic model through categories as an application of algebraic structures, involving issues from order theory, theory of modules and theory of categories in mathematical morphology and image compression. We start from a model on the theory of categories and theory of modules over a commutative ring. We think to combine the construction of a category from an object with homomorphism of modules over quantales using as a model the same issue over categories of modules and over the category of homomorphism of modules over rings. Our intention is to open the way the study of properties of some important modules and their homomorphism that are immediate objects in image processing, seeing them as categories on their own not as a simple object of a category. Through this we want to import the properties from the treatment existing now to investigate after other properties deriving from our point of view.

Introduction

In image processing an image of grey-scale is modelled as a set of all functions from the set of all pixels to the unit interval [0, 1], it is a matrix and its elements represent a pixel whose values are in the set {0, . . ., 255} in the case of a 256-bit encoding, those the “grey-level”, where 0 corresponds to black, 255 to white and the other levels are as lighter as they are closer to 255. The set {0, . . ., 255} is normalized by dividing each element by 255. It can be involved the usual operations, the order relations, t-norms and so on.

Operators used in different tasks can be described in terms of homomorphism of modules over quantales and that of the category of modules over quantales. Important modules, on which these operators are defined, can be tracked as one of the objects of the category of modules over

a quantale in their algebraic treatment and not only, it can be interpreted as a construction from a category of another new one starting from a distinct object of the category.
We find the $[0,1]$-modules $[0,1]^{[0,1]}$ and $[0,1]^n$ in Lukasiewicz transform [4] ($\langle [0,1]^{[0,1]}, \vee, 0\rangle$ and $\langle [0,1]^n, \vee, 0\rangle$) that are Q-modules over the commutative quantale $\langle [0,1], \vee, \odot, 0, 1\rangle$ (for short $[0,1]$-module $[0,1]^{[0,1]}$ and $[0,1]$-module $[0,1]^n$, respectively). We show how we can treat them into categories to prepare a way for further ínterpretation through homomorphism's even in factors.

Definitions and important elements

We bring here some definitions and propositions which we use to bring important elements in our treatement.

A *sup-lattice* is a poset $\langle L, \leq\rangle$ which admits arbitrary joins.

Let Q be a non empty set in which is defined a partial order $\leq$ and a multiplication $\cdot : Q \times Q \to Q$ with a unit element e.

A *quantal* is an algebraic structure $Q = (Q, \vee, \cdot, \perp)$ such that:

1. $\langle Q, \vee, \perp\rangle$ is a sup-lattice;
2. $(Q, \cdot)$ is a monoid;
3. $\forall x \in Q, \{y_i\}_{i\in I} \subseteq Q, x \cdot \vee_{i\in I} y_i = \vee_{i\in I}(x \cdot y_i), \vee_{i\in I} y_i \cdot x = \vee_{i\in I}(y_i \cdot x)$.

Q is said to be commutative if so is the multiplication.

Let M be a non empty set in which is defined a partial order $\leq$ and Q is a quantale. It is also defined in M the algebraic operation denoted by $\cdot$ that maps $Q \times M$ to M. This operation is called the left multiplication in M with elements from Q.

A left Q-module is an algebraic structure $(M; \cdot)$ such that:

1. $\langle M, \vee, \perp\rangle$ is a sup-lattice;
2. $\forall (q_1, q_2, m) \in Q^2 \times M, (q_1 \cdot q_2) \cdot m = q_1 \cdot (q_2 \cdot m)$;
3. $\forall q \in Q, \{m_i\}_{i\in I} \subseteq M, q \cdot^M \vee_{i\in I} m_i = {}^M\vee_{i\in I} q \cdot m_i$;
4. $\forall \{q_i\}_{i\in I} \subseteq Q, m \in M, \left({}^Q\vee_{i\in I} q_i\right) \cdot m = {}^M\vee_{i\in I} q_i \cdot m$;
5. $\forall m \in M, e \cdot m = m$.

In a similar way can be defined the right Q-module. Shortly, we can denote the left Q-module (the right Q-module) with ${}_QM, (M_Q)$.
If a Q-module is a left and right Q-module it is simply called a Q-module.

Let M and N be Q-modules.

A homomorphism (*Q-homomophism*) of Q-module M to Q-module N is a map $f: M \to N$ such that:

1. $f(\vee_{i\in I} m_i) = \vee_{i\in I} f(m_i), \forall \{m_i\}_{i\in I} \subseteq M$;
2. $f(q \cdot m) = q \cdot f(m), \forall q \in Q, m \in M_1, (f(q \cdot m) = q \cdot f(m), \forall q \in Q, m \in M_1)$.

The set of all Q-homomorphism from M to N is denoted by $Hom_Q(M, N)$.

Let $h_i, i \in I$ be maps from Q-module M to Q-module N.

Union of $h_i, i \in I$, denoted by $\sqcup_{i\in I} h_i$, is called a map $\sqcup_{i\in I} h_i : M \to N$, defined by $(\sqcup_{i\in I} h_i)(x) = \vee_{i\in I} h_i(x)$, for all $x \in M$.

Proposition 1. *If* $h_i \in Hom_Q(M,N), i \in I$, *than* $\sqcup_{i\in I} h_i$ *is a* Q*-homomorphism of* M *in* *N*. [5]

In a similar way can be defined the right multiplication $f \cdot q$.

Proposition 2. *If the map* $\in Hom_Q(M,N)$, *and* Q *is a commutative quantale, than* $q \cdot f$ $(f \cdot q) \in Hom_Q(M,N)$, *for all* $q \in Q$.[5]

Associating to all the tuples $(q,f) \in Q \times Hom_Q(M,N)$, *when* Q *is a commutative quantale, the homomorphism* $q \cdot f \in Hom_Q(M,N)$, *we have*

$\cdot: Q \times Hom_Q(M,N) \to Hom_Q(M,N)$,

that is called the left multiplication in $Hom_Q(M,N)$ *with elements from commutative quantale* Q.

Proposition 3. *If a quantal* Q *is commutative and* M, N *are* Q*-modules, than the algebra* $(Hom_Q(M,N); +,\cdot)$ *is a left* Q*-module (right* Q*-module).* [5]

Category of quantale-modules and categories from a quantale-module

Let Q be a quantale and let we considerate the set of all modules over the quantale Q.

By the properties of quantale-modules we see that, for the Q-module M_1, M_2

1) exist $Hom_Q(M_1,M_2)$ of all Q-homomorphism from M_1 to M_2;

2) for all tupples (M_1,M_2,M_3) of Q-modules and for all pairs of Q-homomorphisms $f \epsilon Hom_Q(M_1,M_2)$, $g \epsilon Hom_Q(M_2,M_3)$ is defined their multiplication (the composition) $g \cdot f$ in $Hom_Q(M_1,M_3)$, as a composition of Q-homomorphisms (a usual map composition).

In these conditions, we have:

- $\forall (M_1,M_2),(M_1',M_2') \in \mathcal{C} \times \mathcal{C}$;

$$(M_1,M_2) \neq (M_1',M_2') \Rightarrow Hom_{\mathcal{C}}(M_1,M_2) \cap Hom_{\mathcal{C}}(M_1',M_2') = \emptyset$$
$$\left(Hom_{\mathcal{C}}(M_1,M_2) \neq Hom_{\mathcal{C}}(M_1',M_2')\right);$$

- (*associative property*) $\forall f \epsilon Hom_{\mathcal{C}}(M_1,M_2), \forall g \epsilon Hom_{\mathcal{C}}(M_2,M_3), \forall h \epsilon Hom_{\mathcal{C}}(M_3,M_4)$, $(hg)f = h(gf)$;
- (*identity element*) $\forall M_1 \in \mathcal{C}$ *there exists* $I_{M_1} \in Hom_{\mathcal{C}}(M_1,M_1)$ *such that* $\forall f \epsilon Hom_{\mathcal{C}}(M_1,M_2), \forall g \epsilon Hom_{\mathcal{C}}(M_2,M_1)$, $f \cdot I_{M_1} = f$ *and* $I_{M_1} \cdot g = g$.

As a result of this the set of all Q-modules is a category.

Given a quantale Q, we denote $\mathcal{M}_Q^l$ and $\mathcal{M}_Q^r$ the categories whose objects are, respectively, the left Q-modules and the right Q-modules, and whose morphisms are the Q-module homomorphisms. If Q is a commutative quantale, than $\mathcal{M}_Q^l$ and $\mathcal{M}_Q^r$ are the same, so we denote this category by $\mathcal{M}_Q$.

Given a category [3], by every object of it, we can construct a new one.

With an object M of the category $\mathcal{M}_Q$ we can construct a new one $\mathcal{M}_{Q_M}$. So, the objects are the sets of morphemes $Hom_{\mathcal{M}_Q}(M,X)$ between M and X in $\mathcal{M}_Q$, and the morphisms from $Hom_{\mathcal{M}_Q}(M,X)$ in $Hom_{\mathcal{M}_Q}(M,Y)$ are the morphism $h: X \to Y$, from $Hom_{\mathcal{M}_Q}(X,Y)$, in $\mathcal{M}_Q$ such that, for all $f: M \to X$ and for all $g: M \to Y$, the diagram in figure 3 is commutative

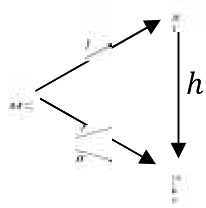

Fig. 1. A morphism h in $\mathcal{M}_{Q_M}$

We see that, the set of morphemes $Hom_{\mathcal{M}_Q}(M,X)$, in $\mathcal{M}_Q$ is the set $Hom_Q(M,X)$ of Q-homomorphisms from M to X. So, the object $Hom_{\mathcal{M}_Q}(M,X)$ of the category $\mathcal{M}_{Q_M}$ is the set $Hom_Q(M,X)$. The set of morphisms from the object $Hom_Q(M,X)$ to the object $Hom_Q(M,Y)$ in $\mathcal{M}_{Q_M}$ is $Hom_{\mathcal{M}_{Q_M}}(Hom_Q(M,X), Hom_Q(M,Y))$.

If we take $[0,1]^{[0,1]}$ or $[0,1]^n$ instead of the Q-module M above we can construct these respective categories:

$\mathcal{M}_{[0,1]_{[0,1]^{[0,1]}}}$ and $\mathcal{M}_{[0,1]_{[0,1]^n}}$ for $\mathcal{M}_{Q_M}$.

The objects of $\mathcal{M}_{[0,1]_{[0,1]^{[0,1]}}}$ are $Hom_{[0,1]}([0,1]^{[0,1]}, N)$, the set of $[0,1]$-homomorphisms from the $[0,1]$-module $[0,1]^{[0,1]}$ to a $[0,1]$-module N.

The morphemes froms $Hom_{[0,1]}([0,1]^{[0,1]}, X)$ to $Hom_{[0,1]}([0,1]^{[0,1]}, Y)$ are the homomorphisms $h: X \longrightarrow Y$ such that the diagram in figure 3 is commutative.

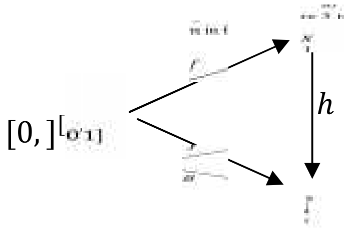

Fig. 3. A morphism h in $\mathcal{M}_{[0,1]_{[0,1]^{[0,1]}}}$

The objects of $\mathcal{M}_{[0,1]_{[0,1]^n}}$ are $Hom_{[0,1]}([0,1]^n, X)$, the set of $[0,1]$-homomorphisms from the $[0,1]$-module $[0,1]^n$ to a $[0,1]$-module X, and in a similar way can be described the morphisms.

A connection between important quantal modules

Let $f: X \to Y$ be a Q-module morphism of $X, Y \in Ob\boldsymbol{Mod_Q}$ and let $Hom_Q(M,X)$ and $Hom_Q(M,Y)$ for a fixed M. We can considerate the map $F_M(f): h \mapsto f \cdot h, \forall h \in Hom_Q(M,X)$.

It is clear that for the homomorphism $F_M(f)$:

1. $F_M(I_X): h \mapsto I_X \cdot h = h \Rightarrow F_M(I_X) = 1_{F_M(X)}$

2. $F_M(g \cdot f)(h) = (g \cdot f) \cdot h = g \cdot (f \cdot h)$

$$= (F_M(g))(f \cdot h) = (F_M(g))\big((F_M(f))(h)\big)$$

$$= \big((F_M(g)) \cdot (F_M(f))\big)(h) \Rightarrow$$

$$F_M(g \cdot f) = F_M(g) \cdot F_M(f).$$

Acordind to the theory of categories and functors [3], we can use F_M as a functor between the category $\boldsymbol{Mod_Q}$ and the category of abelian groups where we find objects of type $Hom_Q(M, X)$ for a fixed M and $X \in Ob\boldsymbol{Mod_Q}$.

Conclusion

In this treatment we operate through theory of categories and we argue that there are categories of quantale modules and with an distinct object in it we can bring a new category. We can operate with factors and use instead of quantal module M the $[0,1]$-modules $[0,1]^{[0,1]}$ and $[0,1]^n$, which play important role in compression and reconstruction of digital image [2][4], with this we want to direct our intention toward other interpretations in image processing.

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