Receding contact problem for two elastic layers resting on a Winkler foundation

Erdal Öner[1], Murat Yaylacı[2], Ahmet Birinci[3]

[1] Department of Civil Engineering, Bayburt University, Turkey
[2]Department of Naval Architecture and Marine Engineering, Karadeniz Technical University, Turkey
[3]Department of Civil Engineering, Karadeniz Technical University, Turkey

Abstract

In this study, receding contact problem for two elastic layers resting on a Winkler foundation and loaded by means of a rigid circular punch is considered. The elastic layers have different heights and elastic constants. External load is applied to the upper elastic layer by means of a rigid circular punch and the lower elastic layer rests on a Winkler foundation. The problem is solved under the assumptions that the contact between elastic layers, and between the rigid punch and the upper elastic layer are frictionless and the effect of gravity forces is neglected. Since the contact between two bodies is assumed to be frictionless, then only compressive normal tractions can be transmitted in the contact areas. General equations of stresses and displacements which are required for the solution of the problem are obtained by using the theory of elasticity and the integral transform techniques. Using integral transform technique and boundary conditions of the problem, the problem is reduced to a system of singular integral equations in which the contact stresses and areas are the unknown functions. The system of singular integral equations is solved numerically by making use of appropriate Gauss-Chebyshev integration formulas and an iterative scheme is employed to obtain the correct contact half-areas that satisfies the equilibrium conditions. Numerical results for the contact stresses and the contact areas are given for various dimensionless quantities.

Introduction

Contact problems have a long history in solid mechanics and have been subject of many investigations due to their practical interest such as foundation grillages, pavements of highway and airfield, railway ballast, rolling mills. A contact problem is named as receding contact if the contact zone shrinks as the bodies are deformed [1]. Alternatively, it is possible to define a problem for which the contact surface in the loaded configuration is contained within the initial contact surface [2]. There is a large body of literature associated with receding contact problems both numerically and analytically. Among the analytical studies on receding contact problems, the following studies can be seen in the literature. The smooth receding contact between an elastic layer and a half space when two bodies are pressed together by considering both plane and axisymmetric cases was considered by [3]. The smooth receding contact problem between an elastic layer and a half space when the layer is compressed by a frictionless semi-infinite elastic cylinder was examined by [4]. A frictionless contact problem for two elastic layers supported by a Winkler foundation was considered by [5]. A frictionless receding contact problem between an anisotropic elastic layer and an anisotropic elastic half plane when the two bodies are pressed together by a rigid circular stamp was examined by [6]. Rhimi at al. [7-8] solved the ax symmetric problem of a frictionless receding contact between an elastic functionally graded layer and a homogeneous half-space when the two bodies are pressed together and a double receding contact axisymmetric problem between a functionally graded layer and a homogeneous substrate. Bagault at al. [9] investigated contact analyses for anisotropic half-space coated with an anisotropic layer.

This paper presents a receding contact problem for two elastic layers resting on a Winkler foundation and loaded by means of a rigid circular punch. It is assumed that friction and gravity forces are neglected. General equations of stresses and displacements which are required for the solution of the problem are obtained by using the theory of elasticity and the integral transform techniques. Using integral transform technique and boundary conditions of the problem, the problem is reduced to a system of singular integral equations in which the contact stresses and areas under rigid circular punch and between elastic layers are the unknown functions. The system of singular integral equations is solved numerically by making use of appropriate Gauss-Chebyshev integration formulas. Contact stresses and the contact areas are obtained for various dimensionless quantities and shown in graphics and tables.

Formulation of the problem and solving the system of integral equations

Consider two elastic layers with different elastic constant and height, resting on a Winkler foundation and subjected to a concentrated load with magnitude P by means of a rigid circular punch as shown in Figure 1. Thickness in z direction is taken to be unit. Since x=0 is the symmetry plane, it is sufficient to consider the problem in the region ($0 \le x < \infty$) only.

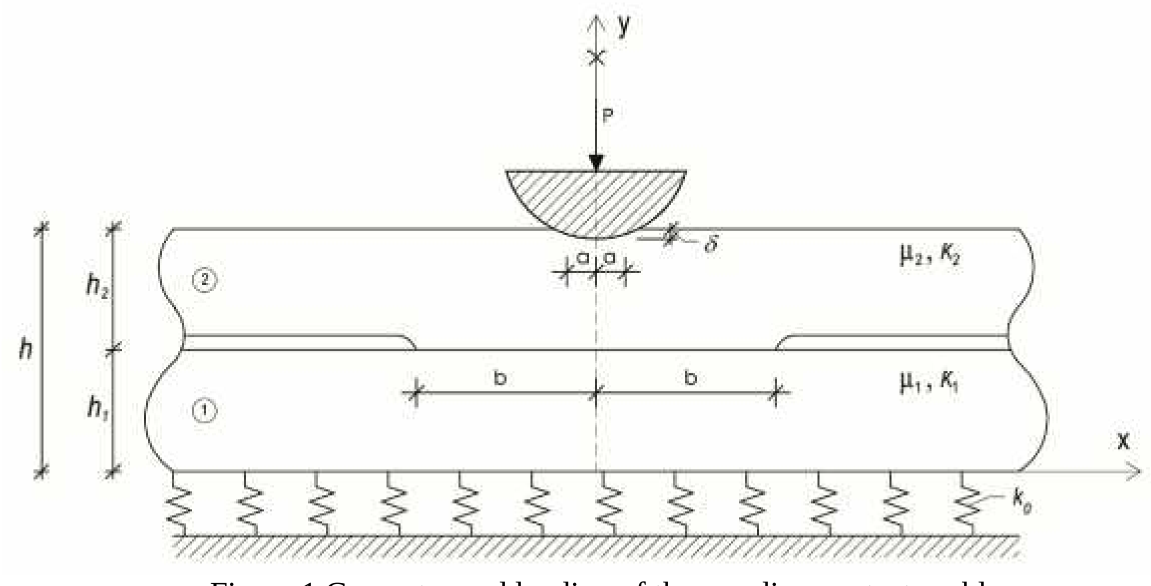

Figure 1 Geometry and loading of the receding contact problem

The stress and the displacement expressions of the layers are obtained using theory of elasticity and integral transform technique as [10].

The boundary conditions of the problem can be written as:

$$\sigma_{y_2}(x,h) = \begin{cases} -p_1(x) & ; \quad (0 \le x < a) \\ 0 & ; \quad (a < x < \infty) \end{cases} \tag{1a}$$

$$\tau_{xy_2}(x,h) = 0 \qquad (0 \le x < \infty) \tag{1b}$$

$$\sigma_{y_2}(x,h_1)=\begin{cases}-p_2(x) & (0\le x<b)\\ 0 \quad ; & (b<x<\infty)\end{cases} \tag{1c}$$

$$\tau_{xy_2}(x,h_1)=0 \qquad (0\le x<\infty) \tag{1d}$$

$$\sigma_{y_1}(x,h_1)=\sigma_{y_2}(x,h_1) \qquad (0\le x<\infty) \tag{1e}$$

$$\tau_{xy_1}(x,h_1)=0 \qquad (0\le x<\infty) \tag{1f}$$

$$\tau_{xy_1}(x,0)=0 \qquad (0\le x<\infty) \tag{1g}$$

$$\sigma_{y_1}(x,0)=k_0 v_1(x,0) \qquad (0\le x<\infty) \tag{1h}$$

$$v_2(x,h)=F(x) \quad \text{or} \quad \frac{\partial}{\partial x}[v_2(x,h)]=f(x); \qquad (0\le x<a) \tag{2a}$$

$$\frac{\partial}{\partial x}\left[v_2(x,h_1)-v_1(x,h_1)\right]=0 \qquad (0\le x<b) \tag{2b}$$

where a is the half-width of the contact area between rigid circular punch and the upper layer, b is the half-width of the contact area between layers. $p_1(x)$ is the unknown contact stress under the rigid circular punch, $p_2(x)$ is the unknown contact stress between two elastic layers. k_0 is coefficient of Winkler foundation. f(x) is the derivative of the function F(x) which characterizes surface profile of the rigid punch. In the case of circular punch profile, f(x) can be obtained as follows:

$$F(x)=h-\delta-\left[\left(R^2-x^2\right)^{1/2}-R\right] \tag{3a}$$

$$f(x)=\frac{d}{dx}\left[F(x)\right]=-\frac{x}{\left(R^2-x^2\right)^{1/2}} \tag{3b}$$

where δ is the maximum displacement which occurs on the layer under the punch on the axis of symmetry (x=0), R is the radius of rigid circular punch. Applying the boundary conditions (1a-1h) to the stress and displacement expressions in [10], A_i, B_i, C_i and D_i (i=1,2) coefficients in [10] can be determined in terms of the unknown contact stresses $p_1(x)$ and $p_2(x)$, and by substituting these coefficients into Eqs. (2a-2b), after some routine manipulations and using the symmetry conditions $p_1(x)=p_1(-x)$ and $p_2(x)=p_2(-x)$, replacing $\omega=\alpha h$ and $r=h_1/h$, the system of integral equations for $p_1(x)$ and $p_2(x)$ is obtained as follows:

$$\frac{1}{\pi}\int_{-a}^{a}\left[\frac{1}{t-x}+k_1(x,t)\right]p_1(t)dt+\frac{1}{\pi}\int_{-b}^{b}\left[k_2(x,t)\right]p_2(t)dt=-\frac{4\mu_2}{(1+\kappa_2)}f(x) \tag{4a}$$

$$\frac{1}{\pi}\int_{-b}^{b}\left[\frac{1}{t-x}+k_3(x,t)\right]p_2(t)dt+\frac{1}{\pi}\int_{-a}^{a}\left[k_4(x,t)\right]p_1(t)dt=0 \tag{4b}$$

where

$$k_1(x,t)=\int_0^\infty\left[\frac{4}{\Delta}K_{11}K_{12}-1\right]\left[\sin\frac{\omega}{h}(t-x)\right]d\omega \tag{5a}$$

$$k_2(x,t) = -8\int_0^\infty \frac{e^{-\omega}e^{-\omega r}}{\Delta} K_{11}K_{13}\left[\sin\frac{\omega}{h}(t-x)\right]d\omega \tag{5b}$$

$$k_3(x,t) = \int_0^\infty \left\{-\frac{4}{(1+m)}\frac{1}{\Delta}\left\{K_{12}\left(-K_{11}\right)+mK_{14}\left[-4\frac{\omega}{h}K_{11A}+K^{**}\left(1+e^{4\omega r}-2e^{2\omega r}\right)\right]\right\}-1\right\} * \left[\sin\frac{\omega}{h}(t-x)\right]d\omega \tag{5c}$$

$$k_4(x,t) = -\frac{8}{(1+m)}\int_0^\infty \frac{e^{-\omega}e^{-\omega r}}{\Delta} K_{11}K_{13}\left[\sin\frac{\omega}{h}(t-x)\right]d\omega \tag{5d}$$

$$\Delta = 4K_{11}K_{14} \tag{5e}$$

$$K_{11} = K^{**}K_{11A} + 4\frac{\omega}{h}K_{11B} \tag{5f}$$

$$K_{12} = e^{-4\omega r} - e^{-4\omega} + e^{-2\omega}e^{-2\omega r}\left(4\omega - 4\omega r\right) \tag{5g}$$

$$K_{13} = e^{-2\omega r}\left(1+\omega-\omega r\right) + e^{-2\omega}\left(-1+\omega-\omega r\right) \tag{5h}$$

$$K_{14} = e^{-4\omega r} + e^{-4\omega} - 2e^{-2\omega}e^{-2\omega r}\left(1+2\omega^2+2\omega^2 r^2-4\omega^2 r\right) \tag{5i}$$

$$K_{11A} = 1-4\omega r e^{2\omega r} - e^{4\omega r} \tag{5j}$$

$$K_{11B} = -1+e^{2\omega r}\left(2+4\omega^2 r^2 - e^{2\omega r}\right) \tag{5k}$$

$$m = \frac{1+\kappa_1}{1+\kappa_2}\frac{\mu_2}{\mu_1} \tag{5l}$$

$$K^{**} = k\left(1+\kappa_1\right) = \frac{k_0}{\mu_1}\left(1+\kappa_1\right) \tag{5m}$$

$$k = \frac{k_0}{\mu_1} \tag{5n}$$

Equilibrium conditions of the problem may be expressed as,

$$\int_{-a}^{a} p_1(t_1)dt_1 = P \quad \text{and} \quad \int_{-b}^{b} p_2(t_2)dt_2 = P \tag{6a-b}$$

In order to simplify solution of the system of integral equations, the following dimensionless quantities are introduced:

$$
\begin{aligned}
&x_1 = ar_1, \quad t_1 = as_1, \quad dt_1 = ads_1 \\
&x_2 = br_2, \quad t_2 = bs_2, \quad dt_2 = bds_2 \\
&g_1(s_1) = \frac{p_1(t_1)}{P/h}, \quad g_2(s_2) = \frac{p_2(t_2)}{P/h} \\
&M_1(r_1, s_1) = k_1(x_1, t_1) \\
&M_2(r_1, s_2) = k_2(x_1, t_2) \\
&M_3(r_2, s_2) = k_3(x_2, t_2) \\
&M_4(r_2, s_1) = k_4(x_2, t_1)
\end{aligned} \tag{7a}
$$

By substituting (7a) into the system of integral equations and equilibrium conditions, the system of integral equations and equilibrium conditions can be obtained as follows:

$$\frac{1}{\pi}\int_{-1}^{1}\left[\frac{1}{s_1 - r_1} + \frac{a}{h}M_1(r_1, s_1)\right]g_1(s_1)ds_1 + \frac{1}{\pi}\frac{b}{h}\int_{-1}^{1}M_2(r_1, s_2)g_2(s_2)ds_2 = -\frac{4\mu_2}{(1+\kappa_2)}f(r_1) \tag{8a}$$

$$\frac{1}{\pi}\int_{-1}^{1}\left[\frac{1}{s_2 - r_2} + \frac{b}{h}M_3(r_2, s_2)\right]g_2(s_2)ds_2 + \frac{1}{\pi}\frac{a}{h}\int_{-1}^{1}M_4(r_2, s_1)g_1(s_1)ds_1 = 0 \tag{8b}$$

$$\frac{a}{h}\int_{-1}^{1}g_1(s_1)ds_1 = 1 \text{ and } \frac{b}{h}\int_{-1}^{1}g_2(s_2)ds_2 = 1 \tag{9a-b}$$

To insure smooth contact at the end points a and b, let the solutions as follows:

$$g_1(s_1) = G_1(s_1)(1 - s_1^{\,2})^{1/2} \qquad -1 < s_1 < 1 \tag{10a}$$

$$g_2(s_2) = G_2(s_2)(1 - s_2^{\,2})^{1/2} \qquad -1 < s_2 < 1 \tag{10b}$$

Using the appropriate Gauss-Chebyshev integration formula [11], Eqs. (8a-b) and (9a-b) become

$$\sum_{k=1}^{N}\frac{1-s_k^{\,2}}{N+1}\left\{G_1(s_{k1})\left[\frac{1}{s_{k1} - r_{i1}} + \frac{a}{h}M_1(r_{i1}, s_{k1})\right] + \frac{b}{h}M_2(r_{i1}, s_{k2})G_2(s_{k2})\right\} = -\frac{4\mu_2}{(1+\kappa_2)}f(r_{i1})$$

(11a)

$$\sum_{k=1}^{N}\frac{1-s_k^{\,2}}{N+1}\left\{G_2(s_{k2})\left[\frac{1}{s_{k2} - r_{i2}} + \frac{b}{h}M_3(r_{i2}, s_{k2})\right] + \frac{a}{h}M_4(r_{i2}, s_{k1})G_1(s_{k1})\right\} = 0$$

(11b)

$$\frac{a}{h}\sum_{k=1}^{N}\frac{1-s_k^{\,2}}{N+1}G_1(s_{k1}) = \frac{1}{\pi} \qquad \text{and} \qquad \frac{b}{h}\sum_{k=1}^{N}\frac{1-s_k^{\,2}}{N+1}G_2(s_{k2}) = \frac{1}{\pi}$$

(12a-b)

where

$$s_k = \cos\left(\frac{k\pi}{N+1}\right), \qquad (k = 1, \ldots, N) \tag{13a}$$

$$r_i = \cos\left(\frac{2i-1}{N+1}\frac{\pi}{2}\right), \qquad (i = 1,..., N+1) \qquad (13b)$$

The system of algebraic equations in (11) and (12) contain 2N + 4 equations for 2N + 2 unknowns. It has been shown that the extra equations in (11a) and (11b) correspond to the consistency conditions of the original integral Eqs. (8a) and (8b). It may also be shown that the (N/2+1) equations in (11a) and (11b) are automatically satisfied. Thus, the equations given by (11) and (12) constitute a system of 2N+ 2 equations for 2N+ 2 unknowns which are $G_1(s_{k1})$, $G_2(s_{k2})$, (k = 1,…,N) and contact half-widths a and b. Note that the system is linear in $G_1(s_{k1})$ and $G_2(s_{k2})$ but highly nonlinear in a and b. Therefore, an iterative procedure had to be used to determine these two unknowns. Making an initial estimate of the variables a and b, the system of equations given by (11) is solved for the 2N unknowns $G_1(s_{k1})$, $G_2(s_{k2})$, (k= 1,…,N). Then, Eq. (12) is used to verify if the equilibrium conditions are satisfied. Since the applied load is known, right-hand terms of Eq. (12) is always constant and left-hand terms of these equation vary from one iteration to another. Based on the physics of the problem, if the right-hand term of (12) is larger in absolute value than left-hand term, the values of a and b are increased or vice versa [12].

Results

Some of calculated results obtained from the solution of receding contact problem described in the previous sections for various dimensionless quantities such as R/h, $\mu_2/(P/h)$, μ_2/μ_1 and k ($k = k_0/\mu_1$) are shown in Tables 1and 2 and Figures 2-5.

Table 1 shows variation of the half-widths of the contact areas (a/h and b/h) with $\mu_2/(P/h)$ and R/h. In the event of increase load ratio $\mu_2/(P/h)$, it is indicated that half-widths of the contact areas a/h and b/h decrease. On the contrary, in the event of increase R/h they increase. Variation of the half-widths of the contact areas (a/h and b/h) with μ_2/μ_1 and k is given in Table 2. It is clearly seen from Table 2 that a/h and b/h increase with increasing of μ_2/μ_1. But they decrease with increasing of k.

Table 1 Variation of the half-widths of the contact areas (a/h and b/h) with $\mu_2/(P/h)$ and R/h ($\chi_1 = \chi_2 = \chi_3 = 2$, $h_1/h=0.5$, $\mu_2/\mu_1=0.5$, k=5)

	R/h=10		**R/h=250**		**R/h=750**	
$\mu_2/(P/h)$	**a/h**	**b/h**	**a/h**	**b/h**	**a/h**	**b/h**
10	0.7389	0.8799	2.0495	2.9998	3.3970	4.9812
50	0.3461	0.4887	1.2820	1.5690	1.7371	2.2283
250	0.1412	0.4701	0.7249	0.9960	1.0921	1.5873
750	0.0805	0.4599	0.4350	0.7241	0.7215	1.1092

Table 2 Variation of the half-widths of the contact areas (a/h and b/h) with k and μ_2/μ_1 ($\chi_1 = \chi_2 = \chi_3 = 2$, $h_1/h=0.5$, R/h=500, $\mu_2/(P/h)=250$)

	k=10		k=4		k=0.1	
μ_2/μ_1	**a/h**	**b/h**	**a/h**	**b/h**	**a/h**	**b/h**
0.1	0.8821	1.4923	0.9365	1.5765	1.1196	1.6238
0.5	0.9451	1.9396	1.0263	2.5786	1.3895	2.8238
5	1.2831	2.2951	1.4356	2.7865	1.9643	3.4576
10	1.5092	2.4227	1.6845	2.9946	2.1276	3.5918

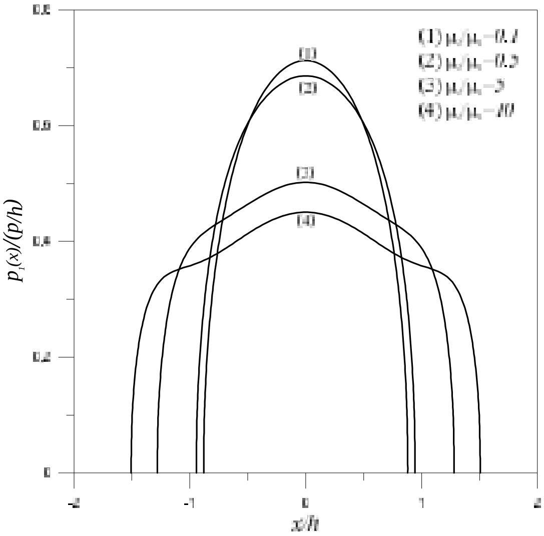

Figure 2 Variation of the contact stress distribution under rigid circular punch with μ_2/μ_1 (h_1/h=0.5, R/h=100, μ_2/(P/h)=250, R/h= 500, k=10)

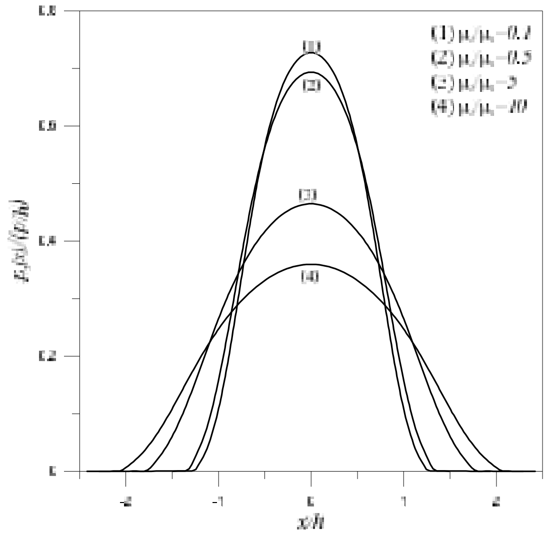

Figure 3 Variation of the contact stress distribution between layers with μ_2/μ_1 (h_1/h=0.5, R/h=100, μ_2/(P/h)=250, R/h= 500, k=10)

Figures 2 and 3 show variations of the contact stress distribution under rigid circular punch and between two elastic layers with μ_2/μ_1. They appear that maximum value of contact stress is always at x=0 and it decreases with increasing μ_2/μ_1. Figures 4 and 5 show variations of the contact stress distribution under rigid circular punch and between two elastic layers with load ratio μ_2/(P/h). As seen in Figs. 4 and 5, $p_1(x)$/(P/h) and $p_2(x)$/(P/h) increase with increasing of μ_2/(P/h)

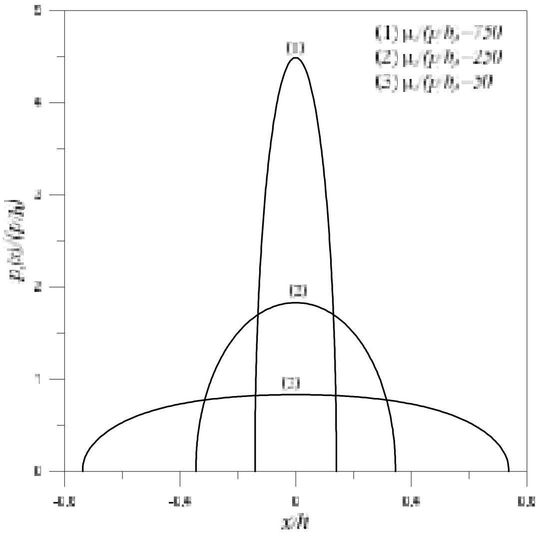

Figure 4 Variation of the contact stress distribution under rigid circular punch with load ratio $\mu_2/(P/h)$ ($h_1/h=0.5$, $R/h=10$, $\mu_2/\mu_1=0.5$, $k=5$)

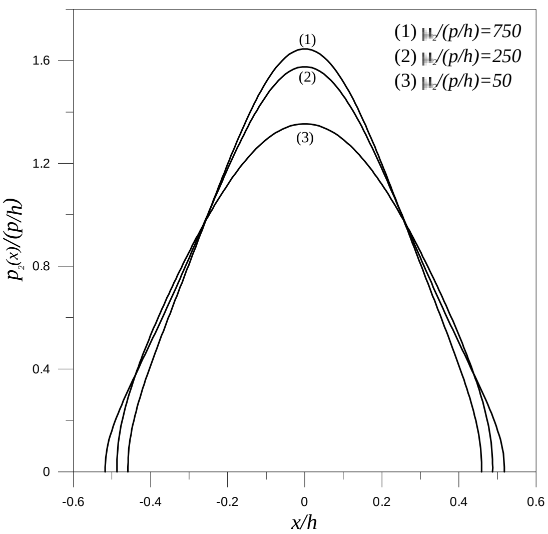

Figure 5 Variation of the contact stress distribution between layers with load ratio $\mu_2/(P/h)$ ($h_1/h=0.5$, $R/h=10$, $\mu_2/\mu_1=0.5$, $k=5$)

Conclusıon

Receding contact problem for two elastic layers loaded by means of a rigid circular punch resting on a Winkler foundation is considered. The results presented in this paper show that elastic properties of the layers, intensity of the applied load, radius of the rigid punch and coefficient of Winkler foundation have considerable effect on at the contact areas and thus, the contact stress distribution under the rigid circular punch and between layers

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