# A stochastic, geostatistic and reliability view on some geotechnical distributions

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## Abstract

In this paper the following problems are treated:

- Estimation of the mean value of a random function Z(x), defined in a stochastic finite element v, (SFE),

$$Z_v=\frac{1}{v}\int_v Z(x)dx$$

where the distributions of Z(x) at each node are known;

- Kriking solution with SFE under the non- stationary hypothesis:

$$E(Z(x))=m(x),\ C(x,h)=E\{(Z(x+h)Z(x)\}-m(x+h)m(x).$$

Finally are given the conclusions underlying the importance of above stochastic instruments not only in the stochastic geotechnical discipline but also in other ones as in energy, geology, geophysics, mechanics, dynamics, elastostatics, finance , engineering , environment, climate etc., in which the distributions are used.

1. Estimation of the mean value of a random function Z(x), defined in a stochastic finite element (SFE) v,

$z_v = 1/v \int_v Z(x)dx$, where the distributions of Z(x) at each node are known;

2. A discretization random field view of SFE in relation to other dicsetized methods.
3. Kriking in SFE view
4. SFE in reliability analysis.
5. Finally some considerations are presented, related to stochastic random field proprieties estimation and stochastic differential equations.

## Introduction

The random theory [1] seeks to model complex patterns of variation and interdependences in cases where deterministic treatments are inefficient and conventional statistics are insufficient

In this view stochastic calculus [2],[4],[13],[14],[15],[16]. on input, output, system models etc are the key points in the random field theory and its applications in geotechnics.

It is known, a random field X(t) is defined as a collection of random variables indexed by a continuous parameter $t \in \Omega \subset Rd$, (d = 2 or 3), i.e. t is a location vector. To completely define a random field, the joint distribution of the random variables {X(t1), X(t2) …, X(tn)} for any {n, t1, t2, …, tn} must be specified.

The continuous random field X(t) as soil , elastic etc medium may be approximated by a discrete , defined in terms of a finite set of random variables {X1, X2,. Xn}. Like in the deterministic FE method, the stochastic discretization method requires the splitting of the domain into a discrete assembly of elements, which is usually referred to as the stochastic finite element (SFE) mesh. All the methods found in the literature under this denomination have the following common characteristics:

– *a finite element model*, i.e. the discretized version of the equations governing a physical phenomenon etc.

– *a probabilistic model* of the input parameters : random variables and/or random fields. In this context below it is presented a SFE method.

**Stochastic finite element. the mean value**

Let's consider: a zone $V \subset R3$ and a random function Z(x) , $x \in V$; the zone V is partitioned into blocks $v_i$ by a parallelepiped grid :

$$V = \bigcup v_i \qquad (1)$$

where $v_i$ is a parallelepiped element with eight nodes. In each node Z(x) is known i.e. the probability density on its point support as it is shown in Fig(1) . It is required:

- the distribution p.d.f [16],[9] in whatever point $x \in V$
- the estimation of the mean value [16,[9]]

$$z_{vi} = 1/v_i \int_v Z(x)dx \qquad (2)$$

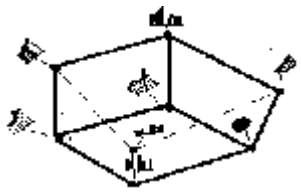

Fig. 1. Parallelpiped element

over the domain $v_i$ .

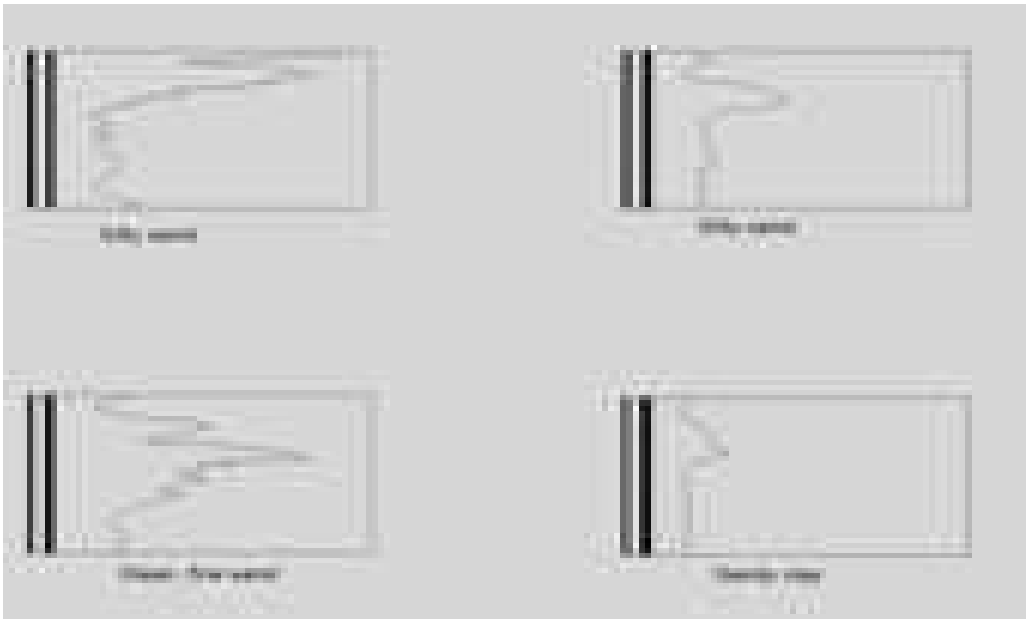

Fig 1a

We define a stochastic element as a block [12], with the random function Z(x), $x \in v_i$, so Z(x) at a point $x \in v_i$ is a random variable with the respective (probability) distributions . In fig 1a are presented four distributions well logs.

Let us consider a reference element $w_i$ in the coordinate system $s_1$ $s_2$ $s_3$. If we choose an incomplete base [16]:

$$P(s) = \langle\ 1\ s_1\ s_2\ s_3\ s_1s_2\ s_2s_3\ \ s_3s_1\ \ s_1s_2s_3\ \rangle \qquad (3)$$

then the function Z(x) could be presented as a linear combination:

$$Z(x)= Z(s_1\ s_2\ s_3)= < P(s) > [\ P_8\ ]^{-1}\ \{\ Z_s^{\ 8}\ \} = <\ N(s)\ >\{\ Z_s^{\ 8}\ \} \qquad (4)$$

$[P_8]^{-1}$ - the matrix, whose elements are the polynomials base values at the nodes,

$\{Z_s^{\ 8}\}$ is the vector of the distributions of the nodes, while $\langle$ N (s) $\rangle$ is the vector of the shape functions $N_i$, i= 1,8

$$N(s)= <\ \ N_i(s) > = < N_1(s)\ , N_2(s)\ldots\ldots\ldots N_8(s) >$$

where

$$Ni(s)= \frac{1}{8}\ (1+s_1s_1^{\ i})\ (1+s_2s_2^{\ i})\ (1+s_3s_3^{\ i})\ \ i=1,8 \qquad (5)$$

To calculate the mean value $z_{vi} = 1/v_i \int_v Z(x)dx$ , we consider the deterministic transformation

$$Xi\ (s)= \ <\ \ N_i(s) > < x_i^{\ 8} > \qquad i=1,3 \qquad (6)$$

Therefore

$$Z_{vi} = \frac{1}{v} \int Z(x_1(s_1\ s_2\ s_3), x_2(s_1\ s_2\ s_3), x_3(s_1\ s_2\ s_3))\ det\ J\ ds_1\ ds_2\ ds_3 = \sum_{1}^{8} H_i Z_i(x)$$

After some algebra [16], we can calculate the weight coefficient Hi, i=1,8, for example

$H_2$=8/3$c_{21}$ $a_{32}$ $c_{13}$ +8/9 $d_{21}$ $a_{32}$ $a_{13}$ +8/27$c_{21}$ $c_{32}$ $d_{13}$+8/27$d_{21}$ $c_{32}$ $b_{13}$ + 8/9 $a_{21}$ $b_{32}$ $c_{13}$ +8/9 $b_{21}$ $b_{32}$ $a_{13}$+8/27$a_{21}$ $d_{32}$ $d_{13}$ +8/27$b_{21}$ $d_{32}$ $b_{13}$ + 8/9 $c_{11}$ $a_{22}$ $d_{33}$+8/9 $d_{11}$ $a_{22}$ $a_{33}$ +8/9 $a_{11}$ $b_{22}$ $c_{33}$ +8/9 $b_{11}$ $b_{22}$ $a_{33}$ + 8/27$c_{11}$ $a_{22}$ $d_{33}$+8/27$d_{11}c_{22}$ $b_{33}$+8/27$a_{11}$ $d_{22}$ $d_{33}$ +8/27$b_{11}$ $d_{22}$ $b_{33}$ + 8/9 $a_{12}$ $a_{23}$ $d_{31}$ +8/9 $b_{21}$ $a_{23}$ $b_{31}$+8/27$c_{12}$ $b_{23}$ $d_{31}$ +8/27$d_{12}$ $b_{23}$ $b_{31}$ + 8/9 $a_{12}$ $c_{32}$ $c_{31}$+8/9 $b_{12}$ $c_{23}$ $a_{31}$ +8/27$a_{21}$ $d_{32}$ $c_{31}$ +8/27$d_{12}$ $d_{23}$ ($a_{31}$ + 8/9 $a_{13}$ $a_{22}$ $d_{31}$+8/9 $c_{13}$ $a_{22}$ $c_{31}$ +8/9 $a_{13}$ $b_{13}$ $b_{31}$+8/9 $c_{13}$ $b_{22}$ $a_{31}$ + 8/9 $b_{13}$ $d_{22}$ $d_{31}$+8/27$d_{13}$ $c_{22}c_{31}$+8/27$b_{13}$ $d_{13}$ $b_{31}$+8/27$d_{13}$ $d_{22}$ $a_{31}$ + 8/9 $c_{21}$ $a_{12}$ $c_{33}$+8/9 $d_{21}$ $a_{12}$ $a_{33}$ + 8/9 $a_{12}$ $b_{12}$ $c_{33}$ +8/9 $b_{21}$ $b_{12}$ $a_{33}$ + 8/27$c_{21}$ $c_{12}$ $d_{33}$+8/27$a_{13}$ $a_{22}$ $d_{31}$+8/27$a_{12}$ $d_{12}$ $d_{33}$+8/27$b_{21}$ $d_{12}$ $b_{33}$ + 8/9 $c_{11}$ $a_{32}$ $c_{23}$+8/9 $d_{11}$ $a_{32}$ $a_{23}$ +8/9 $a_{32}$ $b_{32}$ $c_{23}$ +8/27$b_{11}$ $b_{32}$ $a_{13}$ + 8/9 $c_{11}$ $c_{32}$ $d_{23}$+8/27$d_{11}$ $c_{32}$ $b_{23}$+8/27$a_{32}d_{32}$ $d_{23}$ +8/27$b_{11}d_{32}$ $b_{2\,3}$

($7_c$)

The analogous calculations it could be done [18] for the other coefficient HI, i=1,8 .Also it can easily be prouved

$$\sum_{i=1}^{8} H_i = 1$$

(8)

then $$E\{ Z_v(x)\} = \sum_{i=1}^{8} Hi\ E\ (Zi(x)\ ) = \sum_{i=1}^{8} H_i m = m \quad , \quad ?i=1,8 \qquad (9)$$

where the coefficients Hi are the distribution weights.

It is shown [16],[13] ,the stochastic finite element estimator is a linear interpolator, related to the distributions given at its nodes and it could be calculated using Monte Carlo simulations [6] Generally SFE, there are many applications in geotechnic [5] as in random fields disccretization, stochastic integrals, differential equations(SDE), geostatistics, reliability theory, rock elasticity- plasticity linear theory etc. Of course the mention approach differs to other ones, for example to the midpoint method, the shape function method, the integration point method which use other bases to approximate a given random field.

**Kriking in sfe view**

Kriging refers to a group of generalized least-squares regression algorithms. Kriging refers to a group of generalized least-squares regression algorithms). All kriging methods are variants of the linear regression estimator [3], given by:

$$Z^*(u) = m(u) + \sum_{\alpha=1}^{n(u)} \lambda_\alpha(u)[Z(u_\alpha) - m(u_\alpha)]$$

(10)

where $\lambda_\alpha(u)$ is the weight assigned to datum $z(u_\alpha)$ . m(u) and m($u_\alpha$ ) are the expected values of the random variables defined for each location.

Taking account the the mean value

$$\mathbf{Z_V}=\frac{1}{v}\int_v Z(x)dx \qquad \textbf{(11 )}$$

over a given domain v and the Kriking definition [17],

Above SFE could be used for calculus of Kriking variants [3], [11].simple Kriking, Ordinary krikng, Trend only kriking, Universal kriking, Lognormal kriking Indicator kriking , Cokriking etc

Kriking algorithm with SFE could also be worth for solving stochastic partial differential [6] equations under some conditions

- to approximate" their stochastic coefficients" by mean values in conformity with SFE presented.

- to consider the differential equations in mean square sense,. Remember , a function $f : R^n \rightarrow R$ is called mean square differentiable in $x_0$ in direction x when the differential quotient

$$\lim_{h\to 0}\frac{f(\mathbf{x}_0+h\mathbf{x})-f(\mathbf{x}_0)}{h}$$

has a limit of the mean and the variance for $h\rightarrow 0$. In this view the theorem couuld be considered:

If:

-a random function $f:\mathbb{R}^d\rightarrow\mathbb{R}$ is a mean square differentiable in $x_0$, -

$$\nabla_{\mathbf{x}} f(\mathbf{x}) := \left(\frac{\partial}{\partial x_1}f(\mathbf{x}),\ldots,\frac{\partial}{\partial x_d}f(\mathbf{x})\right)^t \qquad (12)$$

exists and has finite variation

- the trend function $g_1(x)$ , $g_2(x)$ , ……$g_p(x)$ solve the homogeneous equation

$$L_{\mathbf{x}} g_i(\mathbf{x}) = 0 \forall i = 1,\ldots,p \qquad (13_1)$$

2- THE TREND PART $G_0(X)$ SOLVES THE HETEROGENEOUS DIFFERENTIAL EQUATION

$$L_{\mathbf{x}} g_0(\mathbf{x}) = k(\mathbf{x}). \qquad (13_2)$$

- THE COVARIOGRAM EXISTS AND IS ADMISSIBLE , I.E.

$$L_{\mathbf{x}} L_{\mathbf{y}} c(\mathbf{x},\mathbf{y})|_{\mathbf{x}=\mathbf{y}} = 0, \qquad (13_3)$$

for the linear partial differential equation

$$L_{\mathbf{x}} f(\mathbf{x}) = k(\mathbf{x}) \qquad (14)$$

then

its solution is the universal kriking interpolation

$$\hat{f}(x) = g_0(\mathbf{x}) + \begin{pmatrix} f(\mathbf{x}_1) - g_0(\mathbf{x}_1) \\ \vdots \\ f(\mathbf{x}_m) - g_0(\mathbf{x}_m) \\ 0 \\ \vdots \\ 0 \end{pmatrix}^t \cdot \begin{pmatrix} c(\mathbf{x}_1,\mathbf{x}_1) & \cdots & c(\mathbf{x}_1,\mathbf{x}_m) & g_1(x_1) & \cdots & g_p(x_1) \\ \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\ c(\mathbf{x}_m,\mathbf{x}_1) & \cdots & c(\mathbf{x}_m,\mathbf{x}_m) & g_1(x_m) & \cdots & g_p(x_m) \\ g_1(x_1) & \cdots & g_1(x_m) & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ g_p(x_1) & \cdots & g_p(x_m) & 0 & \cdots & 0 \end{pmatrix}^{-1} \cdot \begin{pmatrix} c(\mathbf{x},\mathbf{x}_1) \\ \vdots \\ c(\mathbf{x},\mathbf{x}_m) \\ g_1(\mathbf{x}) \\ \vdots \\ g_p(\mathbf{x}) \end{pmatrix} \tag{15}$$

SFE is a appropriate tool not only to calculate the covariation matrrix of the Kriking or generally in geostatistics , but also the global stiffness matrix **K** of the discretization of randiom fields (soil, mechanic, geotechnic etc systems) , when materials properties are described by means of random variables as Young modulus, Poison ratione etc

,

$$K = \sum_{j=0}^{\infty} K_j \Psi_j \quad \text{where} \quad K_j = \mathrm{E}[K\Psi_j] = \bigcup_e \int_{\Omega_e} B^T \cdot \mathrm{E}[D(\theta)\Psi_j] \cdot B d\Omega_e$$

where

- $\xi = \xi_k(\theta)$ .,k=1,…M) denotes the set of standard normal variables appearing in the discretization of all input random variables and $\{\psi(\xi)\}$ are multidimensional Hermite polynomials, B is the deterministic matrix that relates the component of the strain to the element model displacement and D(θ) is the random elasticity matrix. In case of an isotropic elastic material with random independent Young's modulus E and Poisson's ratio ν the elasticity matrix may be written

$$D = E(\tilde{\lambda}(\nu)D_1 + 2\tilde{\mu}(\nu)D_2)$$

.

where λ(ν) ,μ (ν) are function of ν, which depend on the modeling $D_1$, $D_2$ are detrministic matrices.

Taking into account the our SFE algorithm is independent of integrand form, the proposed SFE could be applied in integrands that appear in geostatistic, SFE methods etc.

**SFE in reliability analysis**.

Structural reliability analysis of large scale mechanical, geotechnical, hydro technical [7], [8] systems require the use of a finite element code coupled with a reliability method. The efficiency of the method in terms of number of calls to the finite element code is crucial when each single finite element run is computationally expensive. In this view we propose our approach as *quadrature method in order to* compute different objects of reliability analysis which include stochastic integrals. To simplify the idea let's consider a mechanical ( geotechnical) model in which we calculate the i-th- moment [7] .

$$\mathrm{E}\left[S^t(X_1, \ldots X_N)\right] = \int_{\mathbb{R}^N} [s(x_1, \ldots, x_N)]^t \varphi(x_1) \ldots \varphi(x_N)\, dx_1 \ldots dx_N$$

where $\boldsymbol{X} = \{X_1, \ldots X_N\}$ is a joint distribution, ψ(.) is the PDF of a standard normal

variable and $s(x_{1,} \ldots\ldots x_N)$ is usually known through a finite element code,, $w_k$ are integration weights. This moment can be estimated :

$$\mathrm{E}\left[S^t(X_1, \ldots X_N)\right] \approx \sum_{k_1=1}^{\mathrm{NP}} \cdots \sum_{k_N=1}^{\mathrm{NP}} w_{k_1} \ldots w_{k_N} [s(x_{k_1}, \ldots, x_{k_N})]^t$$

NP is the order of quadrature scheme. It is seen that $\mathrm{NP}^N$ evaluations of $S$ are needed apriori. also there are an analogy between $w_k$ and $H_k$ of our SFE proposed when the space is $R^3$ analog

## Conclusion

The SFE proposed could be applied for different type of stochastic integrals (i.e. stochastic volume of fluids in porous media etc ), in geostatistic: covariances, moments, different type of krikings, elasticity theory : elasticity, stiffness, damping etc matrices , in reliability theory: discretisation forms , error estimators, second moment approaches, spectral stochastic methods ,stochastic differential equations, random spatial variability etc .In now days , SFE are indispensable in geotechnical plans, studies , designs , projects etc for reliability, risk analysis, stochastic optimization [14],[17],optimal decision making etc

## References

[137] Vanmarcke, E. H. Random field; analysis and synthesis. *The M.I.T, third edition 1983.* USA.

[138] Ghanem, R. Santos,. P. D SFE, a spectral approach. *Springer Verlag*. USA, 1990

[139] Yang Yu. Geostatistical interpolation and simulation of RQD measurement. *(Thesis) The University of British Columbia,* Canada, 2010.

[140] Zhang I, Zhang, L. Tang, W. Kriking numerical models for geotechnical reliability analysis. *Soils and foundations,* **51**(6). Japan, 2011.

[141] Getachew, L. F. Reliability or likelyhood of geological or geotechnical models. Thesis *ITC Delft.* Netherland, 2004.

[142] Gobet, E. Introduction to stochastic calculus and to the resolution of PDE using Monte Carlo simulations *Centre de mathematique appliqués CNRS.* Paris, France, 2012.

[143] Sudret B. Der Kiureghian. Stochastic finite element methods and reliability. *University of California, Bercley.* USA, 2000.

[144] Getachew L. F. Reliability or likelyhood of geological or geotechnical models. Thesis *ITC Delft.* Netherland, 2004.

[145] Jimenez R. Sitar N. The importance of distribution types on finite element analyses of foundation settlement *Computers and Geotechnics 36, Elsevier.* UK, 2009.

[146] Kok. K. P Reliability based design in geotechnical engineering, computations and applicationa. *Routledge, Singapor.* New York, 2008.

[147] Gerald K. Kriking for processes solving partial differential equations. *Freiberg University.* Germany, 2001.

[148] Richard V. Field J. Stochastic model: Theory and simulation.*Sandia National, Laboratories.* USA, 2008.

[149] Hoxha. P, Osmani S. SFE geostatatistcs uncertainty and risk analysis on parameter estimation of rock mechanics.*European Conference on Rock Mechanics and environmentl engineering.* Zurich, Switzerland, 2010.

[150] Osmani S. Optimisation 3. Introduction to stochastics(inAlbanian).*Filara H.* Albania, 2012.

[151] Osmani S. Optimization 2. Finite elements ( in Albanian). *PEGI House.* Albania, 2008.

[152] Osmani S. Energy distribution estimation using stochastic finite element. *Elsevier.Renewable energy Nr.25.* London, UK, 2002.

[153] Osmani. S.Korini Th. Geostatistics ( in Albanian ) Sh.B.L.U. house . Albania 2000

[154] Osmani S. Optimization 1Linear programming with extension.( in Albanian). *PEGI House.* Albania, 2007.