Self-Equilibrium state of V-Expander Tensegrity Beam

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ABSTRACT

In this paper, we study an innovative class of tensegrity beams, obtained by a suitable assembly of elementary V-Expander tensegrity cells along a longitudinal axis in the three-dimensional space.

Tensegrity structures, made by struts in compression and cables in tension, are an innovative structures by itself: they are similar only in appearance to conventional pin-joint structures (trusses), and their mechanics is strongly related to initial feasible self-stress states induced in absence of external loads. In particular, from a kinematical point of view these self-stress states avoid the activation of possible infinitesimal mechanisms.

By a numerical study, we analyze the feasible self-stress states for lightweight tensegrity beams made by a suitable assembly of V-Expander elementary cells. Moreover, we analyze the influence on the feasible self-stress states of the addition of struts or cables starting from the simplest V-Expander configuration.

Keywords: tensegrity beam; self-equibriulium; force density method; numerical methods

INTRODUCTION

Tensegrity structures are an innovative class of lightweight structures, which have gained the interest of researchers in many different fields, including but not limited to engineering. In particular, the interest for tensegrity structures in structural engineering, as well as in architecture, is due to their aesthetic value, their large stiffness-to-mass ratio, the possible deployability, together to their reliability and controllability. The tensegrity concept has found applications within architecture and civil engineering, such as towers, large dome structures, stadium roofs, temporarily structures and tents [1].

Tensegrity is a pin-connected free-standing framework composed of struts in compression and cables necessarily in tension. Usually, the structural analysis of a tensegrity preliminarily requires a form-finding process, since the shape of a tensegrity structure is strictly related to the self-stress in its elements.

In this paper, we show a numerical study of a class of tensegrity beams, obtained by a suitable assembly of elementary V-Expander tensegrity cells along a longitudinal axis in three-dimensional space [2]. The overall performance of this kind of beam is strongly dependent on the way the different elementary cell are connected. By applying a numerical method, we study the self-equilibrium states for the V-Expander tensegrity beam, and we analyze the structural behavior as the pattern of the elements changes.

FORM-FINDING OF TENSEGRITY STRUCTURES

Basic assumptions

Tensegrity structures can be defined as a discontinuous set of elements in compression within a continuous network of tensile elements; this definition is described by well-known expression: "island of compression in an ocean of tension" [3].

The following assumptions are considered in this study: (i) elements are connected by pin joints; (ii) members of the structure are rectilinear; (iii) the connection between the struts is possible only at their extremities; (iv) topology, i.e. the connectivity between the nodes and elements, and the geometrical configuration are known; (v) self-weight of elements is neglected and no external load is applied; (vi) buckling of the strut is not considered; (vii) the structure is free-standing.

From the assumptions, only axial forces are carried by the elements, i.e. there are only two types of elements: struts in compression and cables in tension.

The geometrical configuration of the structure is described in terms of nodal coordinates. Since the structure is free-standing no supports are needed.

A tensegrity structures is a system in a stable self-equilibrated state. The self-equilibrium state refers to the initial mechanical state of the structure before any load, even gravitational, are applied. In this initial state, there is a self-stress state in the elements.

Furthermore, if tensegrity structure possesses any infinitesimal mechanisms, these are stabilized by the self-stress state in the elements. The stability of the structure is defined as the ability of the system to return in equilibrium configuration after a small perturbation [4].

Geometry and topology

In three-dimensional space, a tensegrity structure has *e* elements: *c* cables and *s* struts, ($c + s = e$). The elements of the structure are jointed at *n* nodes. In the cable-net structures, apparently similar to the tensegrity structures, there exist some fixed nodes due to fact that only tension is carried into the cables. Tensegrity structures are free-standing, for assumption, and therefore there exist only free nodes in three-dimensional space. In order to define the geometrical configuration of a tensegrity structures we define $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ ($\in \mathbb{R}^n$) as the nodal coordinate vectors of the free nodes in three directions $\mathbf{e}_x$, $\mathbf{e}_y$, $\mathbf{e}_z$ of an orthogonal reference system $O\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$.

The topology of the tensegrity structures can be defined by a *connectivity matrix* $\mathbf{C}$ ($\in \mathbb{R}^{e \times n}$). If member *k* connects nodes *i* and node *j*, with $i < j$, then in the *k*th row of $\mathbf{C}$ we set 1 and -1 at the *i*th and *j*th position, respectively.

Therefore, the connectivity matrix can be defined as follows:

$$[\mathbf{C}]_{k,p} = \begin{cases} +1 & if\ p = i \\ -1 & if\ p = j \\ 0 & otherwise. \end{cases} \tag{1}$$

Let $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ ($\in \mathbb{R}^e$) the vectors of coordinate differences of elements *k* in *x*, *y*, *z* directions respectively:

$$\begin{cases} \mathbf{u} = \mathbf{Cx} \\ \mathbf{v} = \mathbf{Cy} \\ \mathbf{w} = \mathbf{Cz} \end{cases} \tag{2}$$

and let $\mathbf{l}$ the vector ($\in \mathbb{R}^{e}$) which collects the lengths of the elements. Let $\mathbf{U}$, $\mathbf{V}$, $\mathbf{W}$ and $\mathbf{L}$ ($\in \mathbb{R}^{e \times e}$) be the diagonal form of $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ and $\mathbf{l}$, respectively.

The diagonal matrix $\mathbf{L}$ can be expressed:

$$\mathbf{L}^2 = \mathbf{U}^2 + \mathbf{V}^2 + \mathbf{W}^2 \tag{3}$$

In this way, the geometrical configuration and the topology of the tensegrity structure are completely defined.

Self-Equilibrium state

When geometrical configuration and topology of tensegrity structure are defined, the equilibrium equations in each directions can be set as developed by Scheck [5]. The nonlinear equilibrium equations for unknown locations of the nodes are transformed to a set of linear equations by introducing the so-called force density q_k as the internal axial force to length ratio for each k elements. Note that $q_k > 0$ for cables and $q_k < 0$ for struts.

This condition is associated to the unilateral mechanical behaviour of elements, i.e. cable are in tension, and struts are in compression. In absence of the external loads, the self-equilibrium equations for a general pin-jointed structure can be written as:

$$\begin{cases} \mathbf{C}^T\mathbf{QCx} = \mathbf{0} \\ \mathbf{C}^T\mathbf{QCy} = \mathbf{0} \\ \mathbf{C}^T\mathbf{QCz} = \mathbf{0} \end{cases} \tag{4}$$

where $\mathbf{Q}$ ($\in \mathbb{R}^{e \times e}$) is the diagonal matrix collecting the force densities ratios of all elements. By introducing the *force density matrix* $\mathbf{D}$, ($\in \mathbb{R}^{n \times n}$) as

$$\mathbf{D} = \mathbf{C}^T\mathbf{QC} \tag{5}$$

the equilibrium equations in (4) can be written as

$$\begin{cases} \mathbf{Dx} = \mathbf{0} \\ \mathbf{Dy} = \mathbf{0} \\ \mathbf{Dz} = \mathbf{0} \end{cases} \tag{6}$$

Noting that $diag(\mathbf{b})\mathbf{f} = diag(\mathbf{f})\mathbf{b}$, with $\mathbf{b}$ and $\mathbf{f}$ are a generic vectors, and by introducing the so called *equilibrium matrix* $\mathbf{A}$ ($\in \mathbb{R}^{3n\times e}$), (4) can be rewritten as

$$\begin{bmatrix} \mathbf{C}^T diag(\mathbf{Cx}) \\ \mathbf{C}^T diag(\mathbf{Cy}) \\ \mathbf{C}^T diag(\mathbf{Cz}) \end{bmatrix} \mathbf{q} = \mathbf{0} \tag{7}$$

where the equilibrium matrix is defined as

$$\mathbf{A} = \begin{bmatrix} \mathbf{C}^T diag(\mathbf{Cx}) \\ \mathbf{C}^T diag(\mathbf{Cy}) \\ \mathbf{C}^T diag(\mathbf{Cz}) \end{bmatrix} \tag{8}$$

From the equations in (7), the unknown basis of the vector space of the unknown force densities in the elements lie in the null space of $\mathbf{A}$. The *feasible self-stress state* $\tilde{\mathbf{q}}$, ($\in \mathbb{R}^{e}$) is defined as a state of self-stress that satisfies (7) and should be in accordance with the unilateral behaviour of the elements. In [6] it is shown that $\tilde{\mathbf{q}}$, ($\in \mathbb{R}^{e}$), which can be expressed as a linear combination of the basis of the vector space of the force densities, can be written according to the geometrical symmetry of the structure. In particular, elements in symmetric position have the same force density and then they can be collected in a group.

Let r_A and $\bar{r}_A$, the rank and the dimension of the null space of the equilibrium matrix, hence, there exist $\bar{r}_A$ independent states of self-stress.

$$\bar{r}_A = e - r_A \tag{9}$$

As above mentioned, $\tilde{\mathbf{q}}$ can be written as

$$\bar{\mathbf{q}} = \lambda_1 \mathbf{q}_1 + \lambda_2 \mathbf{q}_2 + ... + \lambda_{\bar{r}_A} \mathbf{q}_{\bar{r}_A} \tag{10}$$

where λ_i, i=1,2,..., $\bar{r}_A$, are real coefficients. Furthermore, let h, the number of groups of the symmetry, vector $\tilde{\mathbf{q}}$ can also be written as

$$\bar{\mathbf{q}} = \mathbf{e}_1 q_1 + \mathbf{e}_2 q_2 + ... + \mathbf{e}_h q_h \tag{11}$$

where $\mathbf{e}_i$ ($\in \mathbb{R}^{e}$), i=1,2,…, h, is a vector composed of a unit in the ith position if the element belongs to the group and zero otherwise. From (10) and (11), a new matrix $\tilde{\mathbf{G}}$ ($\in \mathbb{R}^{e\times(\bar{r}_A+h)}$), and new vector $\bar{\boldsymbol{\beta}}$ ($\in \mathbb{R}^{(\bar{r}_A+h)}$), can be written, which collects the vectors $\mathbf{q}_i$ and $\mathbf{e}_i$, and the real λ_i and q_i, respectively as

$$\tilde{\mathbf{G}} = \left[\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_{\bar{r}_A}, -\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_h \right] \tag{12}$$

$$\bar{\boldsymbol{\beta}} = \left[\lambda_1, \lambda_2, \ldots + \lambda_{\bar{r}_A}, q_1, q_2, \ldots, q_h \right] \tag{13}$$

In this way, in order to find the feasible self-stress state, can be written the follows equation

$$\tilde{\mathbf{G}}\bar{\boldsymbol{\beta}} = \mathbf{0} \tag{14}$$

A Singular Value Decomposition (SVD) should be carried out to find all solutions of (14). These solutions lie in the null space of $\tilde{\mathbf{G}}$. If the dimension of the null space of $\tilde{\mathbf{G}}$ is unit, a tensegrity structure possess a single initial mode of self-stress which is compatible with the unilateral behaviour of the elements, and is in according to the symmetry of the structure in the self-equilibrium.

If the dimension of the null space of $\tilde{\mathbf{G}}$ is equal to zero (14) has only trivial solutions; if the dimension of the null space of $\tilde{\mathbf{G}}$ is more than unit (14) has more than one non-trivial solutions. It is clear that the first $\bar{r}_A$ terms of $\bar{\boldsymbol{\beta}}$ are the real coefficients of the linear combination in (10) and the last *h* terms are the force densities in the groups.

The initial force vector, in the self-equilibrium state, $\bar{\mathbf{f}}_i$ ($\in \mathbb{R}^e$) can be express as

$$\bar{\mathbf{f}} = \mathbf{L}\bar{\mathbf{q}} \tag{15}$$

The Euclidean norm of the vector of unbalanced force $\bar{\mathbf{f}}_u$ ($\in \mathbb{R}^{3n}$) can be used to evaluate the accuracy of the self-equilibrium conditions

$$\bar{\mathbf{f}}_u = \mathbf{A}\bar{\mathbf{q}} \tag{16}$$

Infinitesimal mechanisms

Let $\boldsymbol{\varepsilon}$ ($\boldsymbol{\varepsilon} \in \mathbb{R}^e$), and $\mathbf{d}$ ($\mathbf{d} \in \mathbb{R}^{3n}$) be the vector of the axial strains of the elements and the vector of the nodal displacements, respectively.

By the principle of virtual works

$$\mathbf{A}^T\mathbf{d} = \boldsymbol{\varepsilon} \tag{17}$$

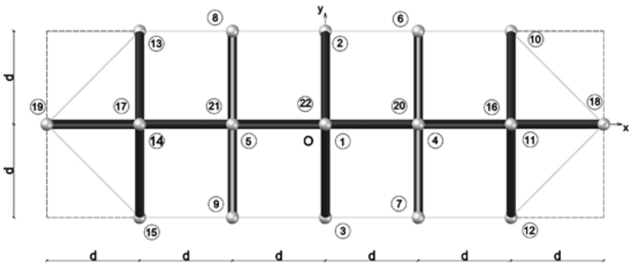
Infinitesimal mechanism $\mathbf{d}_m$ ($\mathbf{d}_m \in \mathbb{R}^{3n}$), is the vector of the nodal displacements which are related to null axial strains, i.e.

$$\mathbf{A}^T\mathbf{d}_m = \mathbf{0} \tag{18}$$

By (18), infinitesimal mechanisms lie in the null space of the transpose of the equilibrium matrix. The dimension of the null space of $\mathbf{A}^T$, which is the multiplicity of infinitesimal mechanisms is

$$\bar{r}_{A^T} = 3n - r_A \tag{19}$$

Rank deficiency conditions



Let r_D the rank of the force density matrix, the dimension of the null space of $\mathbf{D}$ is

$$\bar{r}_D = n - r_D \tag{20}$$

The dimension of the null space of $\mathbf{D}$, in order to create a space of the solution of (6) with at least four dimensions, should be equal or more than four. Furthermore, dimension of the null space of $\mathbf{A}$ should be equal or more than unit. These conditions ensure the possibility to build a non-degenerate self-equilibrated tensegrity structure in three-dimensional space.

V-Expander tensegrity beam

In this paper, we study how the feasible self-stress state $\tilde{\mathbf{q}}$ changes when the number of the elements increases. The V-Expander Tensegrity beam is obtained by assembling three V_{22}-Expander tensegrity cell as shown in Fig. 1. Here, the V-Expander Tensegrity beam is studied as the geometrical parameters *d* (m) and *h* (m) change. Therefore, the feasible self-stress-state is calculated and plotted for *d* = 1 m and *h* = 0.5 m. The first analysed V-Expander beam is composed of sixteen struts and twenty-seven cables; twenty-two nodes connect the elements (nodes and elements are labelled in view of geometrical symmetry of the structure). Then, more complex V-Expander beams are analysed, with the following outline:

- *Case 1*, forty-three elements,
- *Case 2*, fifty-three elements,
- *Case 3*, fifty-seven elements,
- *Case 4*, fifty-nine elements,
- *Case 5*, sixty-three elements,
- *Case 6*, sixty-seven elements,
- *Case 7*, seventy-one elements.

In all the above cases, the tensegrity beam is capable of being enclosed in a paralleliped, the dimensions of which are 6*d*, 2*d* and *h*, in *x*, *y* and *z* direction respectively. For example, in Figure 1 is shown the top view of the V-Expander beam of *Case 1*. In the V-Expander Tensegrity beam, as shown in Fig. 2, referred to *Case 7*, the initial struts are labelled from 48 to 63.

Figure 1 Top view of V-Expander beam in Case 1.

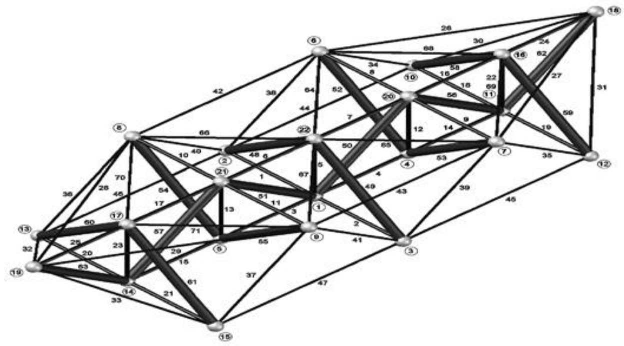


Figure 2 Perspective view of V-Expander beam in Case 7.

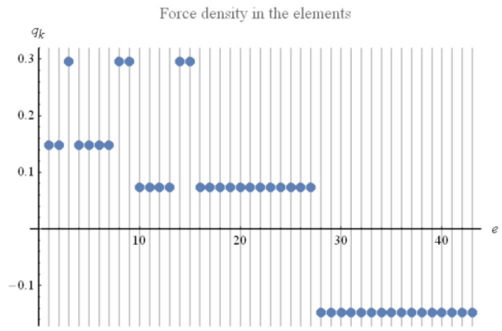
When the number of the elements increases, the rank of the equilibrium matrix **A** increases and, then, the number of the independent self-stress states also increases. Furthermore, the number of the infinitesimal mechanisms of the V-Expander tensegrity beam decreases. In particular, *Case 1* possesses eighteen infinitesimal mechanisms when the six rigid-body motions in three-dimensional space are opportunely constrained; *Case 7* does not possess any infinitesimal mechanisms when the rigid-body motions are excluded. The first case analysed, *Case 1*, is composed of:

- Twenty-seven cables (1, 2, 5, 8, 9, 10, 11, 12, 13, 18, 19, 20, 21, 22, 23, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41) and sixteen struts, which are labelled as mentioned above.

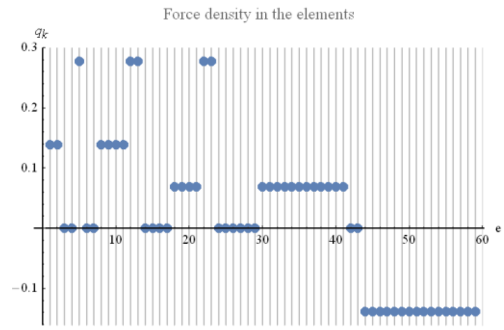
The other cases analysed are obtained, starting form *Case 1*, by addition of the elements (addition respects geometrical symmetry of the structure) as follow:

- *Case 2*, ten elements (3, 4, 6, 7, 24, 25, 26, 27, 28 , 29);
- *Case 3*, four elements (14, 15, 16, 17);
- *Case 4*, two elements (42, 43);
- *Case 5*, four elements (44, 45, 46, 47);
- *Case 6*, four elements (64, 65, 66, 67);
- *Case 7*, four elements (68, 69, 70, 71).

[a]



[d]



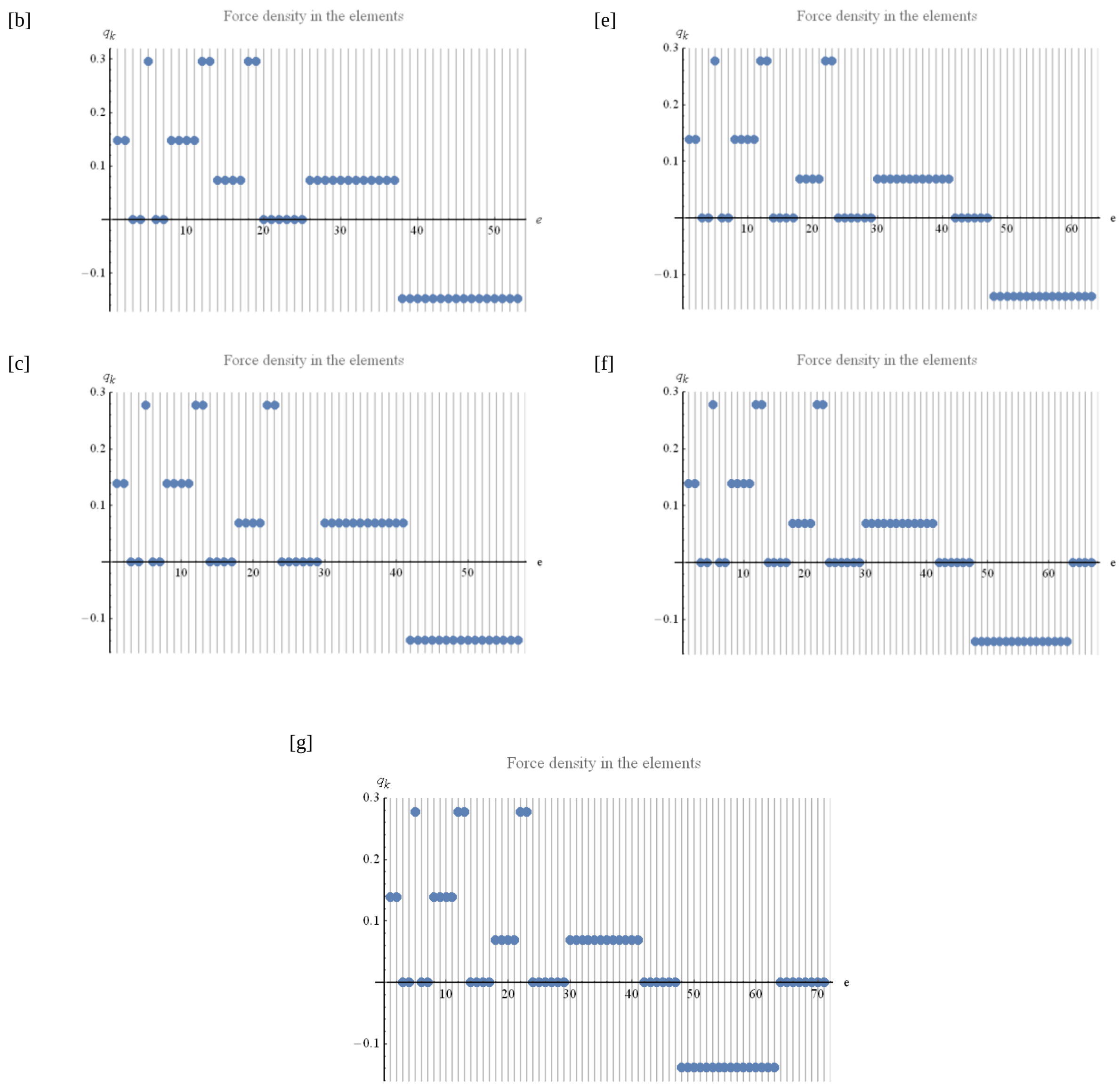


Figure 3 Feasible self-stress states of V-Expander beam [a] Case 1, [b] Case 2, [c] Case 3, [d] Case 4, [e] Case 5, [f] Case 6, [g] Case 7.

In Figure 3 are listed the feasible self-stress states of the seven cases analysed; notice that as the complexity increases, some force densities assume values near to zero. In Figure 4 is shown the number of the independent self-stress states and the number of the infinitesimal mechanisms for every V-Expander beam studied.

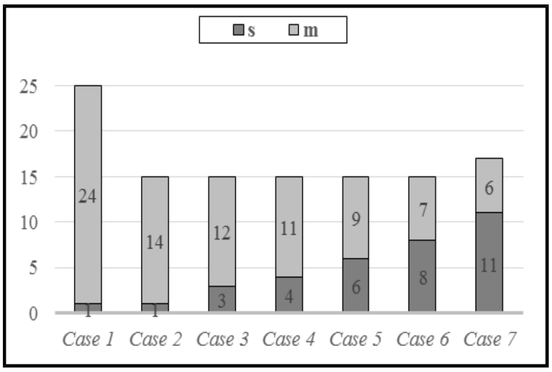


Figure 4 Number of independent self-stress states *s* and number of infinitesimal mechanisms *m* of V-Expander beams.

In Table 1 are listed the properties of the seven V-Expander beams studied.

Table 1 Properties of the seven V-Expander beams.

Case		1	2	3	4	5	6	7
Properties of V-Expander beam	**C** ($\in \mathbb{R}^{e \times n}$)	43 x 22	53 x 22	57 x 22	59 x 22	63 x 22	67 x 22	71 x 22
	A ($\in \mathbb{R}^{3n \times e}$)	66 x 43	66 x 53	66 x 57	66 x 59	66 x 63	66 x 67	66 x 71
	r_A	42	52	54	55	57	59	60

In Table 2 are listed the real coefficients of the linear combination in (10) for *Case 7*.

Table 2 Real coefficients of the linear combination for *Case 7*.

	Case 7										
Real coefficients λ_i *(i=1,2,…,11)*	–0.24499	–0.09704	–0.47821	–0.02461	0.00749	–0.51210	–0.037932	0.06266	0.45970	0.16743	0.27318

CONCLUSIONS

In this paper, we analyse the influence on the feasible self-stress states of the addition of elements (struts and/or cables) starting from an initial configuration of a tensegrity V-Expander beam. In *Case 1* there are possible twenty-four independent infinitesimal mechanisms which lie in the kernel of the transpose of the equilibrium matrix of the structure. Furthermore, in this case, there exists only one independent self-stress state.

From *Case 1* to *Case 7* we observe a redistribution of the force densities in the elements; in particular, we find that the force densities in the vertical cables, i.e. elements 5, 12, 13, 22, 23, decrease from 0.3 to 0.28. Simultaneously, we observe that the force densities in the struts increase until they reach the value -0.138, starting from a value, in *Case 1*, equal to -0.147.

Additional elements "stiffen" the V-Expander tensegrity beam: indeed, disregarding rigid-body motions, the number of infinitesimal mechanisms decreases, and it becomes zero in *Case 7* (the most complex examined beam).

As natural extension of the present work, the mechanical behaviour of the V-Expander tensegrity beam under the action of external loads will be analysed in forthcoming papers.

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