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Double Decomposition Method for the Solution of Sediment Wave Equation

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Abstract *(12 Punto Times New Roman, Bold)*

Transient sediment waves are solved by the double decomposition (DD) method. The method solves the parabolic partial differential equation by decomposing the solution function into summation of *M* number of components. The solution is approximated by considering the first three terms. The performance of the model in simulating experimental data is satisfactory. The hypothetical case study reveals that the model can mimic the sediment transport in nature.

Keywords: *Sediment transport, double decomposition, parabolic partial differential equation*

INTRODUCTION

Sediment transport in alluvial channels has been extensively studied both experimentally and mathematically. Experimental studies have involved extensive flume and field observations (Guy et al. 1966, Soni 1975, Bombar et al 2010, among many). Field studies of sediment transport have also been conducted by many researchers (Langbein and Leopold 1968, Wathen and Hoey 1998, Lisle et al. 2001, among others). Flume experimental and field studies have contributed to the enhanced understanding of basic mechanisms of sediment wave movement in alluvial channels.

Considerable effort has also been devoted to developing theoretical models for predicting sediment movement in alluvial channels. These models have ranged from simple conceptual representations of transport in uniform flow in clear water (de Vries 1973) to comprehensive representations treating transport in sediment laden non-uniform and unsteady flow with interaction between suspended sediment and movable bed layer (Pianese 1994, Wu et al. 2004; Tayfur and Singh 2007).

Analytical solutions have also been attempted for simulating sediment transport in mostly aggrading channels (de Vries 1973; Soni 1981; Dietrich et al. 1999; Shan and Hong, 2001).

The objective of this study is to introduce a mathematical solution technique of the Double Decomposition (DD) method developed by Adomian (1984, 1988) for solving the diffusion equation representing the temporal and spatial change in sediment rate in aggraded channels.

Double Decomposition (DD) Method

The equation to be solved by the DD method is the diffusion equation employed by Soni (1981b):

$$\frac{\partial G}{\partial t} - K_o \frac{\partial^2 G}{\partial x^2} = 0 \qquad (9)$$

$$K_o = \frac{C_o g q_w \left(f/8 \right)^{0.5}}{(1-p)} \qquad (7)$$

Subjected to the following initial and boundary conditions:

$$G\,(x,0) = G_e \qquad (10)$$

$$G\,(0,\,t) = G_e + \Delta G \sin\,(wt) \qquad (11)$$

$$G(L,t) = G_e \qquad (12)$$

where; **G =** sediment transport rate, C_o = coefficient whose value is determined to be 0.372 (Soni 1981b); g = gravitational acceleration; q_w = unit flow discharge; p = porosity, and f = friction coefficient.

The DD method, developed by Adomian (1984, 1988), decomposes the solution function into a sum of components. Thus, the solution function for the transport rate *G(x, t)* can be represented as:

$$G = \sum_{m=0}^{M} G_m \qquad (13)$$

where G_m is the m^{th} approximation of G and $m = 0, 1, 2, 3,\ldots,M$, and M is arbitrary number indicating the order of approximation. In this study, a three-term approximation is considered in this study. Employing the operators, $L_t = \frac{\partial}{\partial t}$ and $L_{xx} = \frac{\partial^2}{\partial x^2}$, Eq. (9) can be written as:

$$L_t(G) = K_o \; L_{xx}(G) \tag{14}$$

Equation (13), with the use of Eq. (14), can be expressed as:

$$\sum_{m=0}^{M} G_m = \sum_{m=0}^{M} W_{x,m} + \frac{1}{K_0} L_{xx}^{-1}\left[L_t \sum_{m=0}^{M} (G_m) \right] \tag{15}$$

where $W_{x,m}$, m = 0, 1, 2, 3,…,M, are coefficients.

The first-term approximation can be expressed as:

$$S_1\,(x,t) = G_0 = W_{\,0,0} + xW_{\,0,1} \tag{16}$$

Applying the first boundary condition (Eq. 11) yields $W_{0,0} = Ge + \Delta G sin\,(wt)$ and second boundary condition (Eq. 12) yields $W_{0,1} = -\Delta G sin(wt)/L$. Substituting these coefficients into Eq. 16 yields the first-term approximation solution as:

$$S_1(x,t) = G_0 = G_e + \Delta G \;\; \sin(wt)\left(1 - \frac{x}{L}\right) \tag{17}$$

The two-term approximation can be expressed as:

$$S_2(x,\, t) = S_1(x,\, t) + G_1 \tag{18}$$

where

$$G_1 = W_{0,1} + xW_{1,1} + \frac{1}{K_o} L_{xx}^{-1}\left[L_t G_o \right] \tag{19}$$

$$L_t G_0 = \Delta G w Cos(wt)\left(1 - \frac{x}{L}\right) \tag{20}$$

$$L_{xx}^{-1}L_t G_0 = \iint \Delta G w \cos(wt)\left(1-\frac{x}{L}\right)dx\,dx = \Delta G\ wCos(wt)\left(\frac{x^2}{2}-\frac{x^3}{6L}\right) \tag{21}$$

Substituting Eq. (21) into Eq. (19) yields $G_1 = W_{0,1} + xW_{1,1} + \frac{1}{K_o}[\Delta G\ wCos(wt)\left(\frac{x^2}{2}-\frac{x^3}{6L}\right)]$.

Substitution of this G_1 into Eq. (18) results in:

$$\mathrm{S_2(x,t)} = G_e + \Delta G\ \sin(wt)\left(1-\frac{x}{L}\right) + W_{0,1} + xW_{1,1} + \frac{1}{K_o}\ \Delta G\ wCos(wt)\left(\frac{x^2}{2}-\frac{x^3}{6L}\right) \tag{22}$$

The use of the first and second boundary conditions (Eqs. 11 and 12) yields $W_{0,1}=0.0$ and $W_{1,1} = -\Delta GwLCos(wt)/(3K_o)$, respectively. Substitution of these coefficients into Eq. (22) results in the second-term approximation as:

$$S_2(x,t) = G_e + \Delta G\ Sin(wt)\left(1-\frac{x}{L}\right) + \frac{\Delta GwCos(wt)}{K_o}\left(\frac{x^2}{2}-\frac{x^3}{6L}-\frac{xL}{3}\right) \tag{23}$$

Note that G_1 is equal to the last term on the right hand side of Eq. (23).

The three-term approximation can be expressed as:

$$S_3(x,\ t) = S_2(x,\ t) + G_2 \tag{24}$$

where

$$G_2 = W_{0,2} + xW_{1,2} + \frac{1}{K_o}L_{xx}^{-1}[L_t G_1] \tag{25}$$

$$L_t G_1 = \frac{\Delta Gw^2 Sin(wt)}{K_o}\left(\frac{xL}{3}-\frac{x^2}{2}+\frac{x^3}{6L}\right) \tag{26}$$

$$L_{xx}^{-1}L_t G_1 = \iint \frac{\Delta Gw^2 Sin(wt)}{K_o}\left(\frac{xL}{3}-\frac{x^2}{2}+\frac{x^3}{6L}\right)dx\,dx = \frac{\Delta G\ w^2 Sin(wt)}{K_o}\left(\frac{Lx^3}{18}-\frac{x^4}{24}+\frac{x^5}{120L}\right) \tag{27}$$

Substituting Eq (27) into Eq (25) yields

$G_2 = W_{0,2} + xW_{1,2} + \frac{1}{K_o}[\frac{\Delta G\, w^2 Sin(wt)}{K_o}\left(\frac{Lx^3}{18} - \frac{x^4}{24} + \frac{x^5}{120L}\right)]$. Substituting this G_2 into Eq (24) results in:

$$S_3(x,t) = G_e + \Delta G\ Sin(wt)\left(1 - \frac{x}{L}\right) + \frac{\Delta G w Cos(wt)}{K_o}\left(\frac{x^2}{2} - \frac{x^3}{6L} - \frac{xL}{3}\right) + W_{0,2} + xW_{1,2} + \frac{\Delta G\, w^2 Sin(wt)}{K_o^2}\left(\frac{Lx^3}{18} - \frac{x^4}{24} + \frac{x^5}{120L}\right) \tag{28}$$

Coefficients in Eq. (28) are found by the application of the first and second boundary conditions (Eqs. 11 and 12) as $W_{0,2} = 0.0$ and $W_{1,2} = -\Delta G w^2 L^3 sin(wt)/(45K_o^2)$, respectively. Substitution of these coefficients into Eq. (28) results in the three-term approximation as:

$$S_3(x,t) = G = G_e + \Delta G\ Sin(wt)\left(1 - \frac{x}{L}\right) + \frac{\Delta G w Cos(wt)}{K_o}\left(\frac{x^2}{2} - \frac{x^3}{6L} - \frac{xL}{3}\right) + \frac{\Delta G w^2 Sin(wt)}{K_o^2}\left(\frac{Lx^3}{18} - \frac{x^4}{24} + \frac{x^5}{120L} - \frac{xL^3}{45}\right) \tag{29}$$

Equation (29) is the double-decomposition solution for the partial differential equation, expressed by Eq. (9). Hence, by Eq. (29), one can obtain the solution for spatial and temporal variation of sediment rate in the channel as a result of sinusoidal loading at the upstream end. The rate of deposition can be obtained by the use of the sediment continuity equation:

$$(1-p)\frac{\partial z}{\partial t} + \frac{\partial G}{\partial x} = 0 \tag{30}$$

where z is the bed level.

Differentiating Eq. (29) with respect to *x* once, and then integrating the resulting expression from t_1 to t_2, one can obtain the following expression for computing temporal and spatial variation of the bed level in a channel as a result of excess sinusoidal loading at the upstream end:

$$z_{t2} = z_{t1} + \frac{1}{1-p}\left[\frac{\Delta G}{Lw}\left(Cos(wt_1) - Cos(wt_2)\right) + \frac{\Delta G}{K_0}\left(Sin(wt_1) - Sin(wt_2)\right)\left(x - \frac{x^2}{2L} - \frac{L}{3}\right)\right]$$

$$\frac{1}{(1-p)}\left[\frac{\Delta Gw}{K_o^2}\left(Cos(wt_2) - Cos(wt_1)\right)\left(\frac{x^2 L}{6} - \frac{x^3}{6} + \frac{x^4}{24L} - \frac{L^3}{45}\right)\right] \quad (31)$$

From Eq. (31), one can obtain the change in bed level in a period of $\Delta t = t_2 - t_1$ at any section of the channel length as a result of the sediment movement in channel due to excess sediment loading at the upstream end of the channel.

MODEL APPLICATION

Experimental Data

The DD solution (Eq. 31) is tested against the experimental data of aggradation depths measured by Soni (1981) in laboratory flume experiments. The flume of rectangular cross-section was 30.0 m long, 0.20 m wide and 0.50 m deep. The flume was filled with sand to a depth of 15 cm, which was the equilibrium bed level depth. The sand forming the equilibrium bed and the injected excess sediment had a median sieve diameter of $d_{50} = 0.32\ mm$ and a specific gravity of 2.65 g/cm^3. The sediment was dropped manually at the upstream section at a constant rate in excess of the equilibrium concentration to cause aggradation. Aggradation runs were continued until the end-point of the transient profiles reached the downstream end. The aggradation runs were conducted using two flow discharges of 4 and 7 *l*/s, slopes ranging from 0.00212 to 0.00652, and with an overloading varying from *0.3Ge* to *4.0Ge*, where *Ge* is the average equilibrium sediment rate. The details of the experiments can be obtained in Soni (1981).

Soni (1981) loaded a constant excess sediment rate during each experiment. Since the DD solution requires a sinusoidal loading the excess loading is represented by a half sinusoidal curve while keeping the total amount constant (Fig.1). For example, for $Ge = 1.66x10^{-5}\ m^2/s$, the excess load of 1.35 Ge = $2.24x10^{-5}\ m^2/s$. That means, every second, there is an excess loading of $2.24x10^{-5}\ m^3/m$ volume of sediment at the upstream end. In 120 minutes, for example, the total loading becomes 0.1614 m^3/m. This excess volume is loaded into the DD solution in 120 min by representing it as a half sine curve as presented in Fig. 1. According to Fig. 1, the period $w = \pi/7200$ and hence loading is zero at $t = 0$ and $t = 7200$ seconds and it reaches a maximum at $t = 3600$ seconds, where the loading is equal to ΔG. ΔG

can be obtained from $\int_0^{7200} \Delta G \; Sin(wt) \; dt \;\; = \;\; 0.1614 \;\; m^3/m$ which results in $\Delta G = 3.65x10^{-5}$ m^3/m.

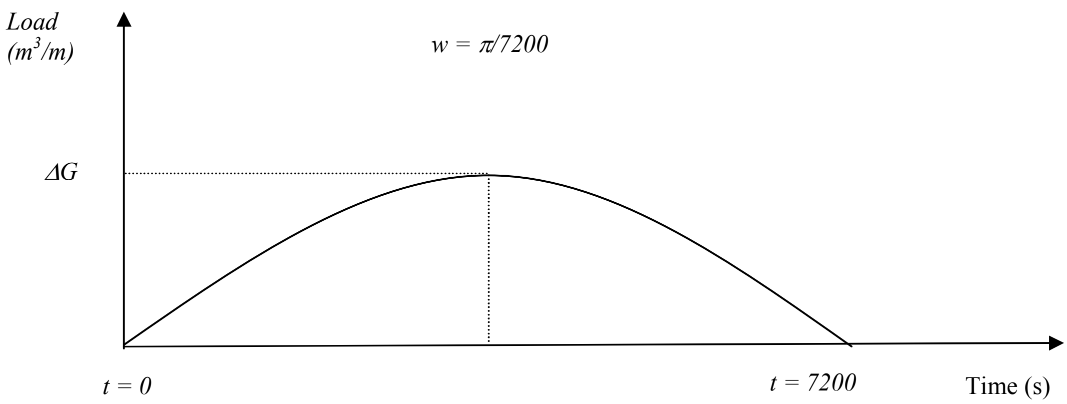

Fig. 1 Schematic representation of a half-sine function for sediment loading

Figures 2—5 present the simulation of bed profiles at 15, 45, 75, and 105 minutes of the experiment, respectively, by the DD solution for the case where excess loading is equal to $1.35\ Ge$, where $Ge = 16.6x10^{-6}\ m^2/s$ (Soni 1975, 1981a, 1981b). The porosity is assumed to be 0.45. From the available information, one can find the shear velocity, $u_* = \sqrt{gRS_o}$ is equal to 0.032 m/s, where R is the hydraulic radius. Then, from $f = \frac{8u_*^2}{u^2}$, one can obtain the value of the friction coefficient as $f = 0.174$, where u is the flow velocity. Then, from $K_o = \frac{C_o g q_w \left(f/8 \right)^{0.5}}{(1-p)}$, one can obtain the value of $K_o = 0.02$, which is in agreement with Soni (1981b).

Also, shown in those figures are comparisons against the error-function solution (Eq. 3) and the numerical solution (Tayfur and Singh 2006). Note that the error function solution employs $G(\infty, t) = Ge$, i.e., as the channel is sufficiently long, the aggradation profile would reach the original bed level. This is analogous to the boundary conditions which are often employed in groundwater flow and contaminant transport problems. The numerical solution solves the system of flow continuity and momentum, and sediment continuity equations. It approximates the momentum equation by the kinematic wave approximation, i.e., it ignores the convective and local acceleration terms in the momentum equation. For sediment transport function, it employs the kinematic wave theory model developed by Tayfur and Singh (2006). It also employs Dietrich (1982)'s formulation for particle fall velocity and Bridge and Dominic (1984)'s model for particle velocity. It relates suspended sediment concentration to flow variables and particle characteristics through Velikanov (1954)'s relation. It solves the system of equations by the Lax explicit finite difference method. The details of the numerical model can be obtained in Tayfur and Singh (2006).

Figure 2 shows the simulation at the 15th minute of the experiment. The measured level reaches the equilibrium bed at the 3rd meter, and then fluctuates significantly. The sediment wave front reaches the equilibrium level earlier at the 2nd m in the case of the numerical model, but at the 5th m in DD solution. In the case of error-function solution, it never reaches the original level. Fig. 3 shows the bed profile simulation at the 45th minute. The measured profile reaches the equilibrium bed level around the 10th meter, although it fluctuates from that location onward. The numerical model and DD solutions closely follow the measured profile until the 10th meter. The error-function solution, although, on average, follows the measured profile, in the first 10 meters, it over-predicts. However, after the 10th meter, it shows better performance than others in capturing the measured bed levels. Fig. 4 shows the simulation at the 75th minute of the experiment. As seen, the measured profile gradually reaches the original level at around the 15th meter, while the numerical model reaches that level at the 12th meter, and DD model at the 10th meter, both closely following the measured profile. The error-function solution reaches the original level at a later distance of the 18th meter. It significantly over-predicts the measured data, especially in the first 14 meter-distance. Figure 5 shows the simulation at the 105th minute of the experiment. The sediment wave front reaches the equilibrium bed level at an earlier distance, around the 12th meter in the DD solution. The numerical model shows better performance in capturing the measured data.

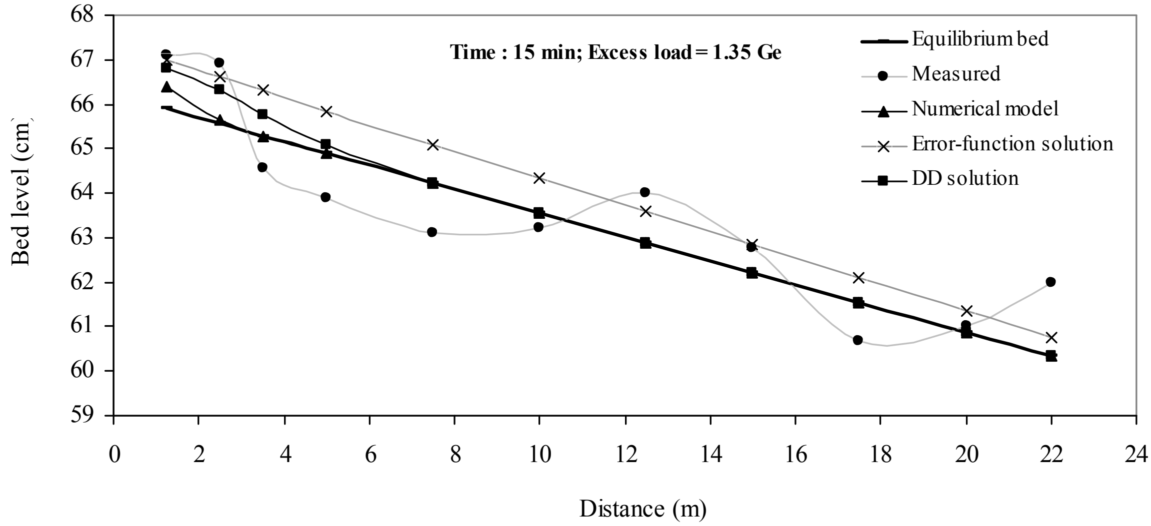

Fig. 2 Simulation of bed profile at 15 min of the experiment

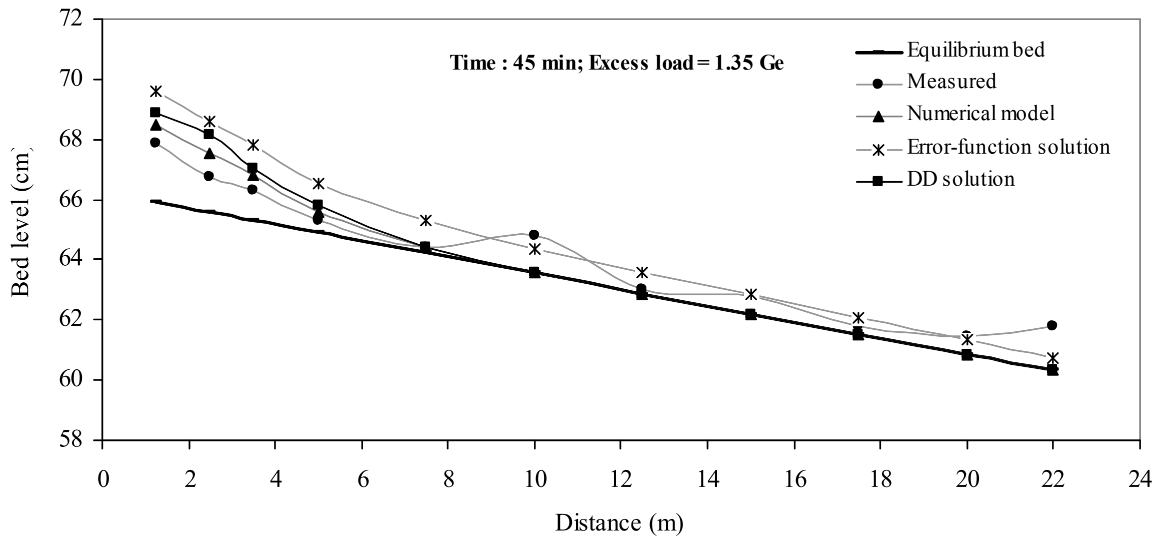

Fig. 3 Simulation of bed profile at 45 min of the experiment

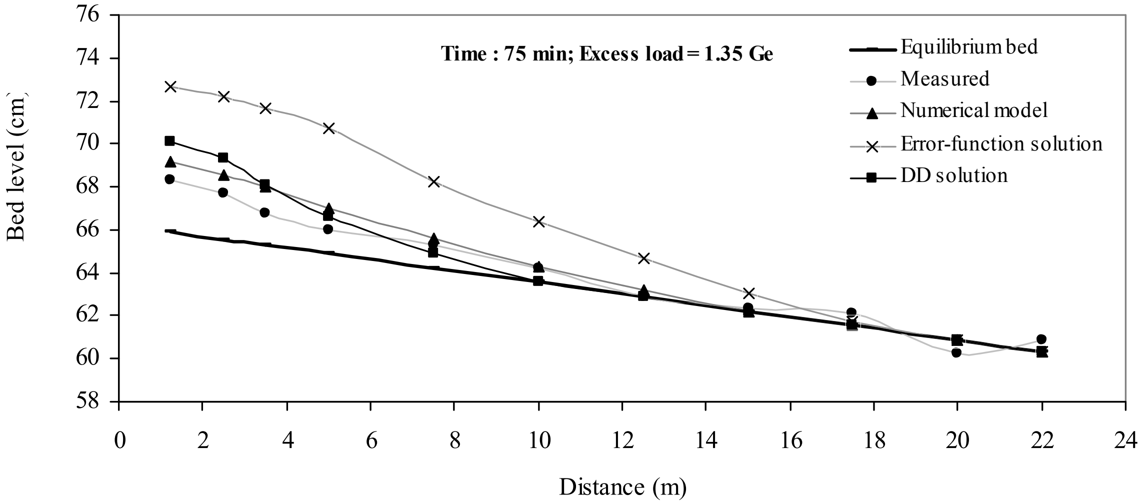

Fig. 4 Simulation of bed profile at 75 min of the experiment

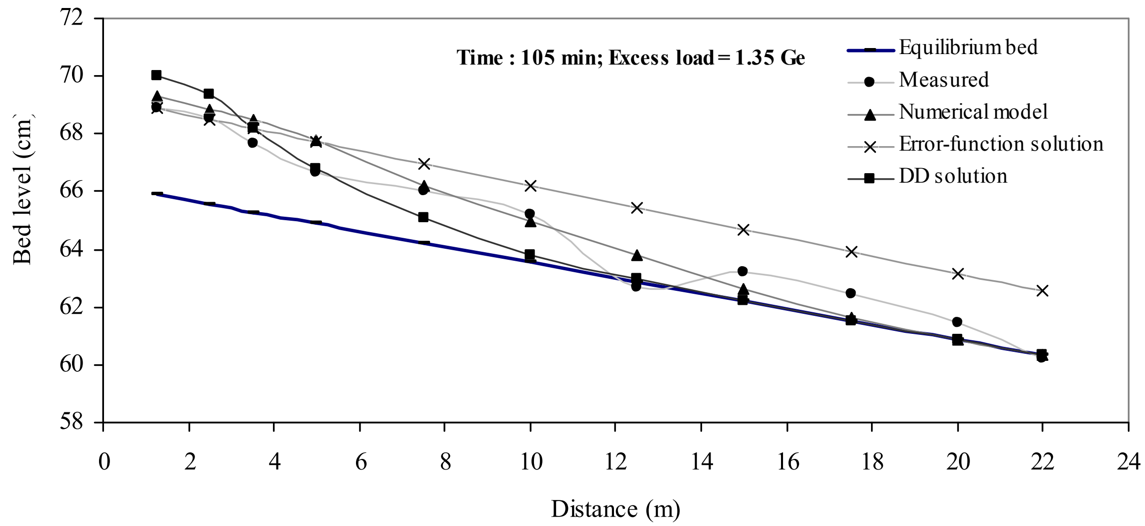

Fig. 5 Simulation of bed profile at 105 min of the experiment

Hypothetical Case Application

The DD solution is investigated for a hypothetical channel having a 200 m length, 20 m width, and bed slope of 0.0025. It is assumed that flow rate $Q = 20\ m^3/s$, *Chezy coefficient* $C_z = 20\ m^{0.5}/s$, *porosity* $p = 0.40$. A constant sediment loading of 10 ton/h is assumed at the upstream end of the channel.

Fig. 9 shows the simulation of bed profiles at different times of the simulation. As time progresses bed level increases along the channel length. Fig. 10 shows the bed profile simulation at the 30th minute, under different sediment loadings. As the loading increases, bed levels increase along the channel length. Fig. 11 shows the simulation of bed profiles along the channel under the same sediment loadings of 10 ton/h but different K_o values. Note that different K_o-values imply different flow conditions. The higher is K_o-value, the higher the flow discharge becomes. Under higher flow discharges, the bed levels are low. This is because a high flow rate carries the sediment faster downstream of the channel, thus resulting in low bed levels. These results imply that the model can produce results compatible with those that one may observe in the field.

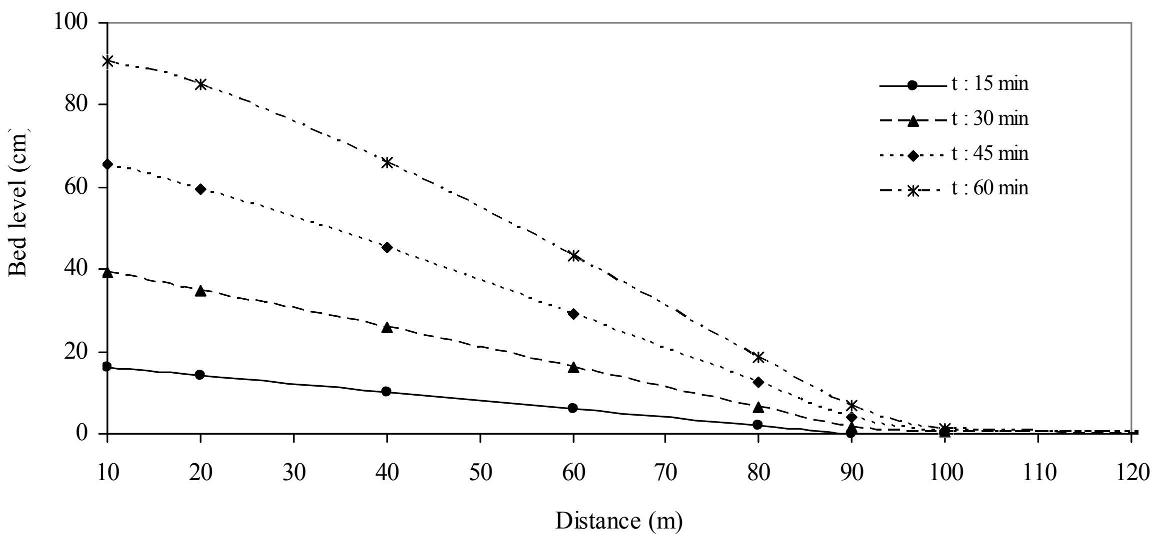

Fig. 9 Simulation of bed profile along the hypothetical channel at different times of the simulation period

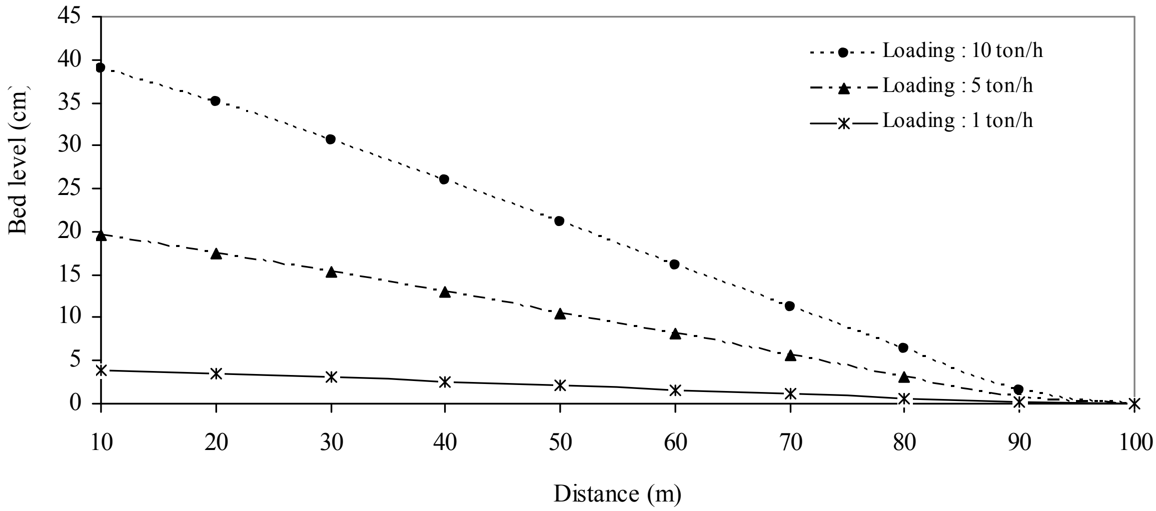

Fig. 10 Simulation of bed profile at the 30th min along the hypothetical channel under different sediment loadings

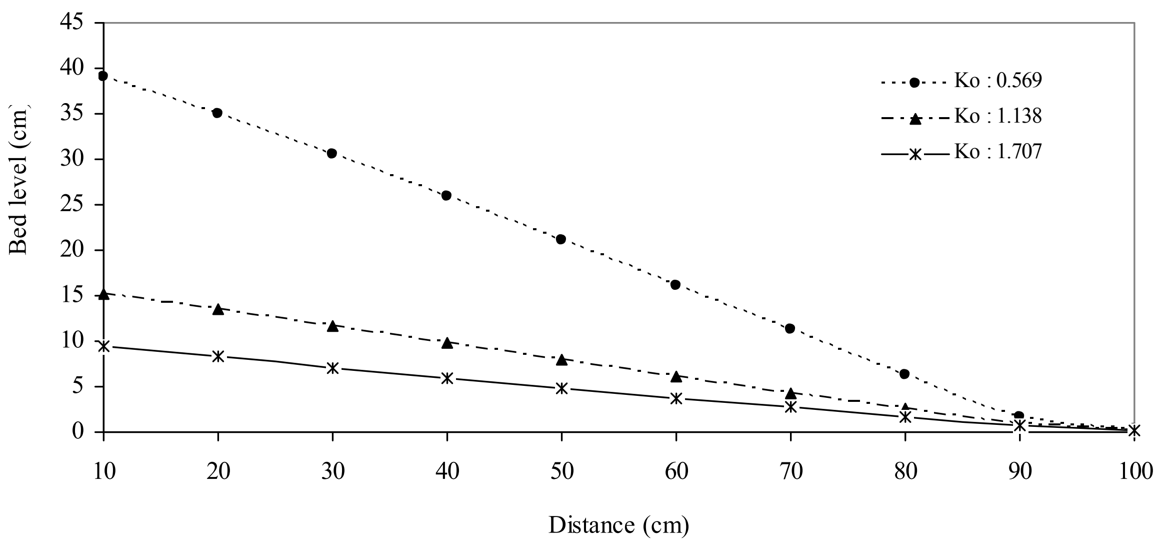

Fig. 11 Simulation of bed profile at the 30th min along the hypothetical channel under different Ko-values

CONCLUDING REMARKS

The satisfactory performance of the solution and the comparison analysis of the order of approximations imply that considering only the first 3 terms of the series solution (3^{rd} order approximation) is sufficient for this particular problem.

The performance of the model against the numerical and analytical (error-function) solutions is satisfactory for simulating the experimental data. It nearly shows the same performance as the numerical model but it is mostly better than the analytical solution. On the contrary to the error-function solution, the developed model does not impose any constraint on the solution, such as the infinite channel length.

The model is not satisfactorily able to predict bed levels after the middle section of the channel at later periods of simulation.

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