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Differential Quadrature Method For Solving Bed Load Sediment Transport

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ABSTRACT

Sediment transport is crucial for designing, and operating hydraulic structures. Hence, its prediction has forced researches to study it through experimental and mathematical modeling works. Mathematical modeling has gained importance especially with the advent of powerful computers. These modeling studies are mostly based on the numerical solutions of transport equations of the partial differential equations with finite difference, finite element or finite volume methods. This study, as an alternative to existing methods, has developed a numerical technique, called differential quadrature method (DQM). The DQM expresses a differential at a point as a function of products of weight coefficients and the functional values at each point of the domain. The weight coefficients are determined using one of the several algorithms such as the Langrangian, depending upon the spacing intervals. In this study, bed load sediment transport equation, coupled with flow equations of the continuity and momentum, is solved using the DQM. The performance of the model is tested against that of the finite difference method and as well as the experimental data. The results revealed that DQM can also be employed in modeling bed load sediment transport.

INTRODUCTION

Sediment transport is a major issue in terms of various problems such as changes in the stream bed, flood control, erosion prevention, water intake, etc. Solving, by using different wave theories, water and sediment continuity equation and the equation of motion, can provide the sediment transport rates and bed profiles. The amount of transported sediment depends on the properties of the sediment and turbulence of the flow. For that reason, the solutions can be done under some assumptions. Generally finite difference, finite elements, or finite volume methods are used in the solutions of these equations. Fuladipanah et al. [1] solved one-dimensional water and sediment equations with the implicit finite difference method for Sandy River. Chiari et al. [2] investigated bed load transport using kinematic wave approach and explicit upwind finite difference scheme in the excessive slope channel. Seo et al. [3] solved suspended sediment transport using a three-dimensional finite elements method.

Employing the kinematic wave theory [4] for the sediment transport function, this study proposes to solve the system of the equations using fairly a new method of differential quadrature [6, 10].

SEDIMENT TRANSPORT MODEL

Sediment transport in rivers consists of two different movements that are suspended sediment transport and bed-load sediment transport. Moving bed is a porous medium thus having porosity. For a wide and rectangular cross-section, sediment transport equations can be written as follows [12]:

$$\frac{\partial h(1-c)}{\partial t}+\frac{\partial hu(1-c)}{\partial x}+p\frac{\partial z}{\partial t}=0 \tag{1}$$

$$\frac{\partial hc}{\partial t}+\frac{\partial huc}{\partial x}+(1-p)\frac{\partial z}{\partial t}+\frac{\partial q_{bs}}{\partial x}=0 \tag{2}$$

In the equations (1) and (2), h: flow depth (L), u: flow velocity (L/T), c: suspended sediment concentration (L^3/L^3), z: bed-load sediment elevation (L), p: porosity (L^3/L^3), and q_{bs}: sediment flux movable bed layer (L^2/T). Equations (1) and (2) are included five unknown variables (h, u, c, z, and q_{bs}). Therefore, three more equations are needed to close the system. First equation can be written using kinematic wave equation for the flow momentum as:

$$u=\alpha h^{\beta} \tag{3}$$

where α and β coefficients can be determined from one of the friction equations. Using the equation of Chezy, $\alpha = C_z\sqrt{S_o}$, $\beta = 0.5$ where C_z is Chezy friction coefficient and S_o is channel bed slope. Volumetric sediment concentration can be determined by Eq (4) using flow parameters [13].

$$c=\frac{\kappa}{g v_f}\frac{u^3}{h} \tag{4}$$

where κ: sediment transport capacity coefficient, g: gravitational acceleration (L/T^2) and v_f: sediment particle fall velocity (L/T).

Different empirical equations are proposed for the sediment transport rate in the moving bed. Following (M/L/T) [5]:

$$q_{st}=v_s C_b\left(1-\frac{C_b}{C_{b\max}}\right) \tag{5}$$

sediment flux can be determined as follows:

$$q_{bs} = q_{st} / \rho_s \tag{6}$$

In these equations, v_s: particle velocity (L/T), C_b: areal sediment concentration (M/L^2); C_{bmax}: maximum areal sediment concentration when transport ceases (M/L^2); ve ρ_s sediment mass density (M/L^3). Furthermore, the areal sediment concentration can be related to the bed level (z) as [12]:

$$C_b = (1-p) z \rho_s \tag{7}$$

Substitution of equations (6) and (7) into equation (5) would result in the following equation relating sediment flux to bed elevation (sediment concentration):

$$q_{bs} = (1-p) v_s z (1 - \frac{z}{z_{\max}}) \tag{8}$$

where z_{max} is the maximum bed elevation (L) [12].

Substituting equation (3) and (4) into equation (1) and (2) yields:

$$(1 - 3\beta\delta\alpha^3 h^{2\beta}) \frac{\partial h}{\partial t} + (\beta\alpha h^{\beta-1} - 4\beta\delta\alpha^4 h^{3\beta}) \frac{\partial h}{\partial x} + p \frac{\partial z}{\partial t} = 0 \tag{9}$$

$$3\beta\delta\alpha^3 h^{2\beta} \frac{\partial h}{\partial t} + 4\beta\delta\alpha^4 h^{3\beta} \frac{\partial h}{\partial x} + (1-p) \frac{\partial z}{\partial t} + \frac{\partial q_{bs}}{\partial x} = 0 \tag{10}$$

Flow and sediment transport are depended on not only flow variables *u* and *h*, but also to the sediment particle velocity. In turn, sediment particle velocity depends on particle and flow properties [14, 15]. Details of these models are given in [12].

DIFFERENTIAL QUADRATURE METHOD AND SOLUTION

In DQM, first proposed Bellman et al. [16], the partial derivative of a function with respect to a variable at a discrete point is approximated as a weighted linear sum of the function values at all discrete points in the region of that variable. The approximation of the partial derivative can be written as:

$$u_x(x_i) = \left.\frac{\partial^r u}{\partial x^r}\right|_{x=xi} = \sum_{j=1}^{N} A_{ij}^{(r)} u(x_j) \qquad i = 1,2,\ldots,N \tag{11}$$

where $u_x^{(r)}$ is the r[th] order derivative of the function, x_j are the discrete points of the variable x, $u(x_j)$ are the function values at points x_j and $A^{(r)}_{ij}$ are the weight coefficients for the r[th] order derivative of the function. Determining the weight coefficients is the most crucial step in use of DQM. Shu and Xue [17] worked on the selection of the weight coefficients and proposed several solutions in their studies. The weight coefficients change upon approximation function and according to the chosen approximation function the method takes different names such as Polynomial Differential Quadrature (PDQ), Fourier Expansion Base Differential Quadrature (FDQ) and Harmonic Differential Quadrature (HDQ) [18, 19]. For boundary value problems, DQM performance is highly dependent on the boundary conditions and sampling grid points. The boundary conditions can be easily implemented to DQ system and the common type of boundary conditions, which are Dirichlet, Neumann and/or mixed type function, do not create any difficulty in this implementation process [20, 21]. The overall sensitivity of the model especially depends on the location and number of sampling grid points. However, Civalek [21] points out that the determination of the effective choice of sampling grid points for any problem reduces the analysis time. For instance, previous studies show that for the solution of linear equations with homogeneous boundary conditions, selecting equal intervals between the adjacent grid points are adequate. On the other hand, for the vibration problems, the choice of grid points through the Chebyshev-Gauss-Lobatto method is more reasonable. In time-bound equations and initial value problems, selection of unequal intervals for sampling grid points produces the appropriate solutions [21].

Grid points are written using Chebyshev-Gauss-Lobatto method as follows.

$$x_i = \frac{1}{2}\left(1 - \cos\frac{i-1}{N-1}\pi\right) \tag{12}$$

For calculation of weight coefficients Lagrange interpolation function was used:

$$L_i = \prod_{j=1}^{N} (x_j - x_i) \tag{13}$$

The weight coefficients are expressed as:

$$\begin{aligned} A_{i,j} &= \frac{L_j}{L_i(x_i - x_j)} && i \neq j \\ A_{i,i} &= -\sum_{j=1}^{N} A_{i,j} && i \neq j \end{aligned} \tag{14}$$

when Eqs (9) and (10) are ordered, equation (15) and (16) are obtained.

$$(1-3\beta\delta\alpha^3 h_i^{2\beta})\frac{\partial h}{\partial t}+(\beta\alpha h_i^{\beta-1}-4\beta\delta\alpha^4 h_i^{3\beta})\sum_{j=1}^{N}A_{i,j}h_i+p\frac{\partial z}{\partial t}=0 \tag{15}$$

$$3\beta\delta\alpha^3 h^{2\beta}\frac{\partial h}{\partial t}+4\beta\delta\alpha^4 h^{3\beta}\sum_{j=1}^{N}A_{i,j}h_i+(1-p)\frac{\partial z}{\partial t}+\sum_{j=1}^{N}A_{i,j}q_{bs,i}=0 \tag{16}$$

From Eqs (15) and (16), with explicit approximation, $h_i^{t+\Delta t}$ and $z_i^{t+\Delta t}$ can be calculated as:

$$h_i^{t+\Delta t}=h_i^t+\left(\frac{\Delta t}{1-3\beta\delta\alpha^3 h_i^{2\beta}-p}\right)^t\left(\left(4\beta\delta\alpha^4 h_i^{3\beta}+(1-p)\beta\alpha h_i^{\beta-1}\right)^t\sum_{j=1}^{N}A_{i,j}h_i^t+p\sum_{j=1}^{N}A_{i,j}q_{bs,i}^t\right) \tag{17}$$

$$z_i^{t+\Delta t}=z_i^t+\frac{1}{p}\left((3\beta\delta\alpha^3 h_i^{2\beta}-1)^t(h_i^{t+\Delta t}-h_i^t)-\Delta t(\beta\alpha h_i^{\beta-1}-4\beta\delta\alpha^4 h_i^{3\beta})^t\sum_{j=1}^{N}A_{i,j}h_i^t\right) \tag{18}$$

MODEL APPLICATION

Laboratory Experiment

The experimental data of Soni [22] are employed. This experiment was carried out in a flume that is 24.5m long, 0.2m wide having rectangular cross section. The flume was filled with sand to a depth of 15 cm. The sand forming the bed and the injected sediment had a median sieve diameter of d_{50} = 0.32 mm and a specific gravity of 2.65 g/cm^3. The discharge was controlled by a valve and measured by a means of calibrated orifice meter installed in the supply line. A pointer gauge mounted on a movable carriage was used to record the bed elevations. Suspended sediment transport was negligible. Other details of experiment are given in Soni [22]. In the numerical solution, suggested by Langbein and Leopold [5], C_{bmax} = 245 kg/m^2 value was used. Porosity value was assumed to be 0.40. z_{max} was calculated from Eq (7) [23].

Measurement and calculated results using DQM, Lax scheme and Mc Cormack scheme for bed profile are presented in Figures 1—3. Error rate variations of numerical solutions are also given Figures 1—3. The computed RMSE (Root mean square error) values are summarised in Table 1.

Results of numerical solution using DQM are close to experimental results and as well as the other numerical methods. Since implementation of DQM is relatively easy, it can be considered as an alternative method to the other numerical methods.

Hypothetical Case

In this case, sediment transport was investigated in rectangular channel having width 20m, length 1000m, slope 0.0025 and Cheesy coefficient 50. Sediment properties are assumed to be: ρ_s= 2650 kg/m^2, d_{50}= 0.32mm and porosity p = 0.528. Flow rate is assumed to be 50 m^3/s, h = 1 m and initial sediment level 1 cm. The assumed inflow hydrograph and sedimentopraph are presented in Figure 4. Calculated bed profiles are shown in Figure 5, and comparative simulations are presented in Figure 6.

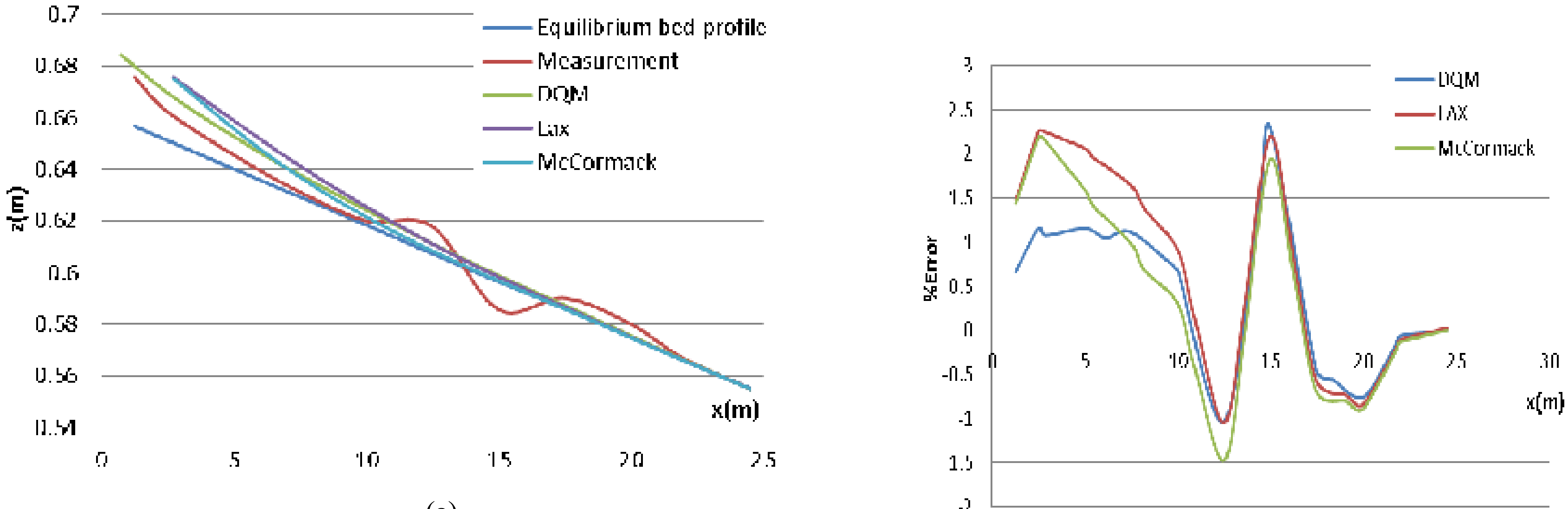
Differences between numerical methods are negligible, as seen in Figure 6. Maximum value of difference is about %0.06.

Table 1 RMS Error values of numerical solutions

	t=30min.	t=60min.	t=90min.
DQM	4.10	4.02	3.73
Lax	5.83	4.95	4.29
McCormack	5.10	5.23	5.36

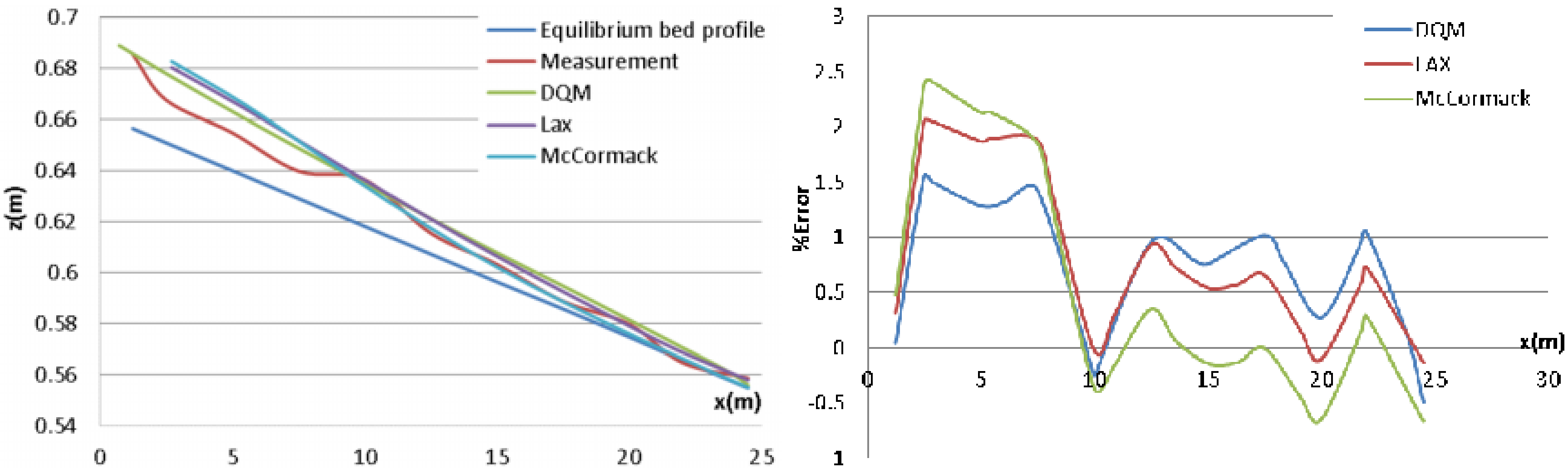
CONCLUSION

In the numerical solutions of sediment transport problems so far, finite difference, finite element, or finite volume methods were generally used. DQM is a method of numerical solution of partial differential equations. In this study, for the first time, in the numerical solution of sediment transport problem, DQM was used. DQM produced results close to experimental and as well as those of the other numerical methods. This may indicate that the DQM can be an alternative method, in addition to the existing ones, for the solution of sediment transport in alluvial channels. Under different boundary and initial conditions, this method can be further tested for more complex problems.



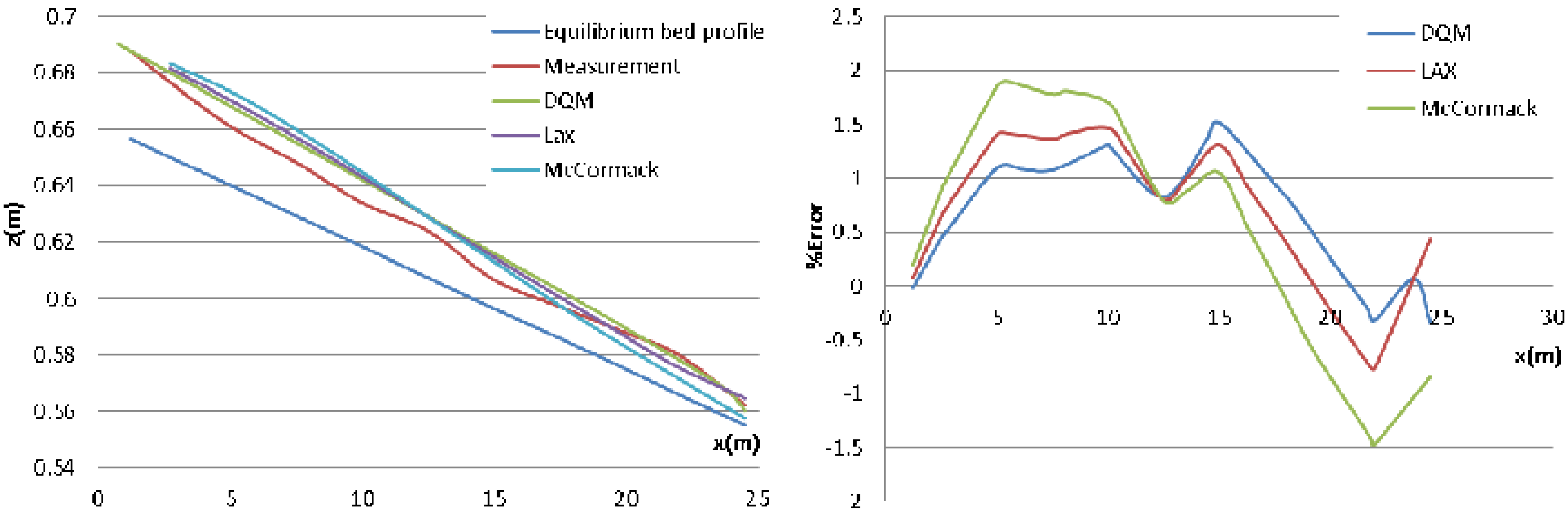
(a)

Figure 1. (a) Measured and calculated bed profiles at t = 30 min. (b) %Error values



(a) (b)

Figure 2. (a) Measured and calculated bed profiles at t = 60 min. (b) %Error values



(a)

(b)

Figure 3. (a) Measured and calculated bed profiles at t = 90 min. (b) %Error values

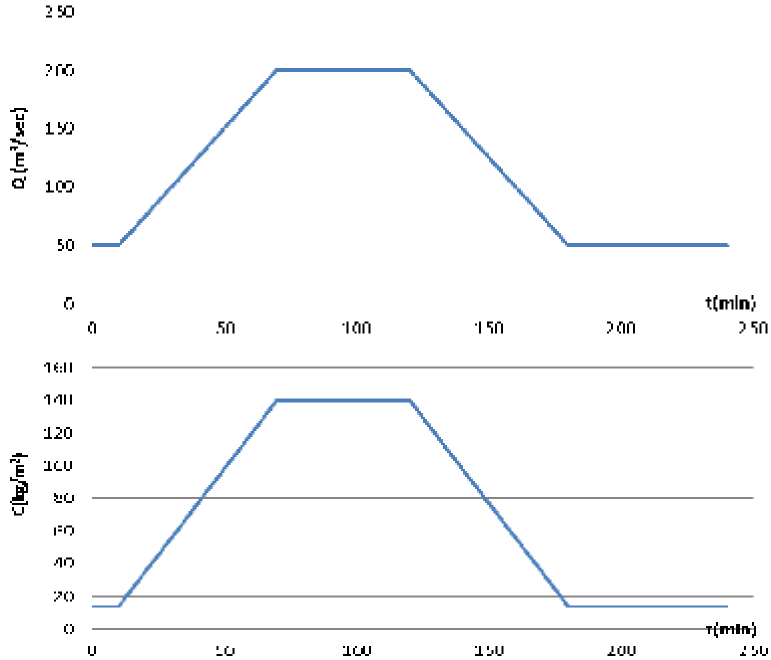


Figure 4. Inflow hydrograph and sedimentograph

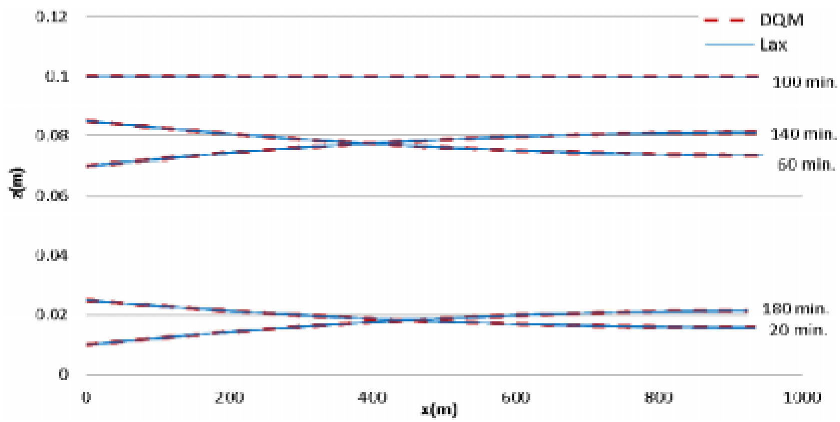


Figure 5 Change of bed profiles

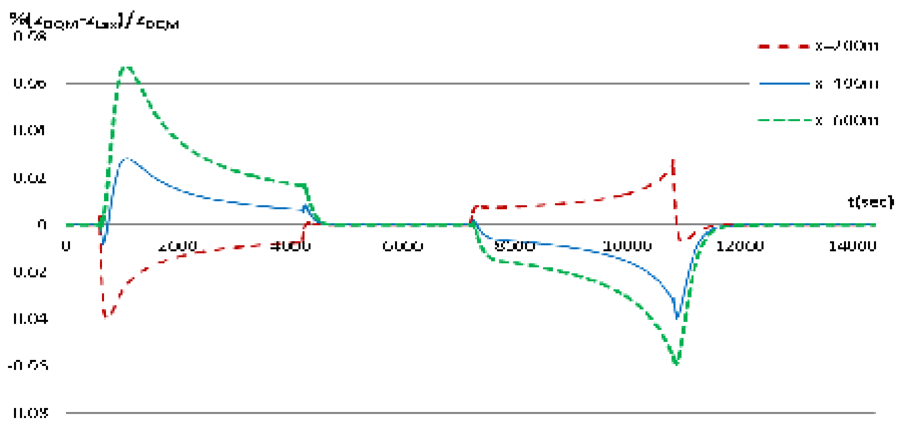


Figure 6 Comparison of DQM against other numerical solutions

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