International Balkans Conferance on Challenges of Civil Engineering, BCCCE,
19-21 May 2011, EPOKA University, Tirana, ALBANIA

Dynamic Analysis Of Straight And Curved Beams On Elastic Foundation

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Abstract

Free and forced vibration analysis of straight and curved beams on elastic foundation are investigated in Laplace domain. The Timoshenko beam theory is adopted in the derivation of the governing equation. The curvature of the rod axis, effect of rotary inertia and, shear and axial deformations are considered in the formulation. Ordinary differential equations in scalar form obtained in the Laplace domain are solved numerically using the complementary functions method to calculate the dynamic stiffness matrix of the problem. The solutions obtained are transformed to the real space using the Durbin's numerical inverse Laplace transform method. The dynamic analysis of straight and curved beams on elastic foundation are analyzed through various examples.

Key Words: Straight and curved beams, Elastic foundation, Free vibration, Forced vibration, Complementary functions method.

1. Introduction

Beams and plates resting on elastic foundation have wide application in engineering practice. The dynamic analysis of beams is investigated using various foundation models. Numerous studies have been performed to investigate the static deflection and dynamic response of the beams resting on various elastic foundations.

Kıral and Ertepınar [1] investigated the isothermal behavior of planar rods resting on an elastic foundation and subjected to a static loading. Kukla [2] and Wang [3] studied the problem of free, lateral vibration of a Bernoulli-Euler beam supported on a step like varying Winkler elastic foundation. Haktanır and Kıral [4] studied the behavior of continuous and elastically supported helicoidal structures by the stiffness matrix approach based on transfer matrix method. De Rosa and Maurizi [5] calculated the exact free vibration frequencies of a Euler beam on two parameter elastic soil. Chen et al. [6] studied bending and free vibration of arbitrarily thick beams on a Pasternak elastic foundation. Çalım [7] investigated dynamic behavior of beams on Pasternak-type viscoelastic foundation subjected to time-dependent loads. Later, Çalım and Akkurt [8] studied static and free vibration analysis of straight and circular beam on elastic foundation.

In this study, the numerical procedure is used to analyze the free and forced vibrations of straight and circular beams on elastic foundation. The curvature of the rod axis, effect of rotary inertia and shear and axial deformations are also considered in the formulation. Ordinary differential equations in scalar form obtained in the Laplace domain are solved numerically using the complementary functions method to calculate the dynamic stiffness matrix of the problem [7-9]. The solutions obtained are transformed to the time domain using the Durbin's numerical inverse Laplace transform method [7-9].

2. The governing equations

Consider a naturally curved and twisted spatial slender rod. The trajectory of geometric centre G of the rod is defined as the rod axis and its position vector at t=0 is given by $\mathbf{r}^o=\mathbf{r}^o(s,\ 0)$ where s is measured from an arbitrary reference point s=0 on the axis. A moving reference frame is defined by the unit vectors **t**, **n**, **b** with the origin on the rod axis, where **t**, **n** and **b** are tangent, normal and binormal vectors, respectively. The following differential relations among the unit vectors **t**, **n**, **b** can be obtained with the aid of the Frenet formulae [10]:

$$\partial\mathbf{t}/\partial s=\chi\mathbf{n}\ ,\ \partial\mathbf{n}/\partial s=\tau\mathbf{b}-\chi\mathbf{t}\ ,\ \ \partial\mathbf{b}/\partial s=-\tau\mathbf{n} \qquad (1)$$

where χ and τ are the curvature and the natural twist of the axis, respectively. For planar rods $\tau = 0$, and for straight rods $\chi = \tau = 0$.

Let the displacement of a point on the rod axis be $\mathbf{U}^o(s,\ t)$, and the rotation of the cross-section about an axis passing through the geometric centre G be $\mathbf{\Omega}^o(s,\ t)$. Assuming the displacements and the deformations are infinitesimal, and that the material of the rod is homogenous, linear elastic and isotropic the governing equations of a space rod are obtained in vectorial form as:

$$\frac{\partial\mathbf{U}^{O}}{\partial\mathbf{s}}+\mathbf{t}\times\mathbf{\Omega}^{O}-\mathbf{C}^{-1}\,\mathbf{T}^{\mathbf{o}}=0\,,\quad \frac{\partial\mathbf{\Omega}^{O}}{\partial\mathbf{s}}-\mathbf{D}^{-1}\,\mathbf{M}^{\mathbf{o}}=0 \qquad (2)$$

$$\frac{\partial\mathbf{T}^{O}}{\partial\mathbf{s}}+\mathbf{p}^{(\mathbf{ex})}-\mathbf{p}^{(\mathbf{in})}=0\,,\qquad \frac{\partial\mathbf{M}^{O}}{\partial\mathbf{s}}+\mathbf{t}\times\mathbf{T}^{O}+\mathbf{m}^{(\mathbf{ex})}-\mathbf{m}^{(\mathbf{in})}=0 \qquad (3)$$

where the inertia force vector is $\mathbf{T}^0$, the inertia moment vector is $\mathbf{M}^0$ and $\mathbf{p}^{(ex)}$ and $\mathbf{m}^{(ex)}$ are the external distributed load and external distributed moment vectors per unit length of axis, respectively. The mass density ρ, the inertia force $\mathbf{p}^{(in)}$ and the inertia moment $\mathbf{m}^{(in)}$, per unit length of the rod axis are given as

$$p_i^{(in)} = -\rho\, A \frac{\partial^2 U_i^o}{\partial t^2}, \qquad m_i^{(in)} = -\rho\, I_i \frac{\partial^2 \Omega_i^o}{\partial t^2} \qquad (i = t, n, b) \tag{4}$$

C and **D** are defined as

$$[C] = \begin{bmatrix} EA & 0 & 0 \\ 0 & GA/\alpha_n & 0 \\ 0 & 0 & GA/\alpha_b \end{bmatrix} \qquad [D] = \begin{bmatrix} GI_t & 0 & 0 \\ 0 & EI_n & 0 \\ 0 & 0 & EI_b \end{bmatrix} \tag{5}$$

where A is area of cross-section, E and G are elastic constants, α_n and α_b are shear coefficients, I_t is torsional and I_n, I_b are bending moments of inertia.

Let $\mathbf{p}^{(ex)}$ and $\mathbf{m}^{(ex)}$ be composed of two parts such that

$$\mathbf{p}^{(ex)} = \mathbf{p}^e - \mathbf{p}^f, \qquad \mathbf{m}^{(ex)} = \mathbf{m}^e - \mathbf{m}^f \tag{6}$$

where the quantities with superscripts e and f denote, respectively, the loading and the foundation reaction on the beam. $\mathbf{p}^f$ and $\mathbf{m}^f$ are the foundation stimulated force and moment per unit length of the beam,

$$p_i^f = k_i\, U_i^o, \qquad m_i^f = (k_1)_i\, \Omega_i^o \qquad (i = t, n, b) \tag{7}$$

where k and k_1 are the spring constants.

Assuming that the centroid and the shear center of cross-section coincide, the **n**, **b** axes become the principal axes. Moreover, the effect of warping of the cross-section is neglected. The system of 12 ordinary differential equations governing the dynamic analysis of the curved beams on elastic foundation with respect to the moving coordinate system, is obtained in canonical form in the Laplace domain as

$$\frac{d\overline{U}_t}{ds} = \chi\, \overline{U}_n + \frac{1}{EA}\overline{T}_t \tag{8a}$$

$$\frac{dU_n}{ds} = -\chi\, \overline{U}_t + \tau\, \overline{U}_b + \overline{\Omega}_b + \frac{\alpha_n}{GA}\overline{T}_n \tag{8b}$$

$$\frac{d\overline{U}_b}{ds} = -\tau\, \overline{U}_n - \overline{\Omega}_n + \frac{\alpha_b}{GA}\overline{T}_b \tag{8c}$$

$$\frac{d\overline{\Omega}_t}{ds} = \chi\, \overline{\Omega}_n + \frac{1}{GI_t}\overline{M}_t \tag{8d}$$

$$\frac{d\overline{\Omega}_n}{ds} = -\chi\, \overline{\Omega}_t + \tau\, \overline{\Omega}_b + \frac{1}{EI_n}\overline{M}_n \tag{8e}$$

$$\frac{d\overline{\Omega}_b}{ds} = -\tau\, \overline{\Omega}_n + \frac{1}{EI_b}\overline{M}_b \tag{8f}$$

$$\frac{d\overline{T}_t}{ds} = z^2 \rho A\, \overline{U}_t + \chi\, \overline{T}_n + k_t \overline{U}_t - \overline{p}_t^{(ex)} \tag{8g}$$

$$\frac{d\overline{T}_n}{ds} = z^2 \rho A \overline{U}_n + \tau \overline{T}_b - \chi \overline{T}_t + k_n \overline{U}_n - \overline{p}_n^{(ex)} \tag{8h}$$

$$\frac{d\overline{T}_b}{ds} = z^2 \rho A \overline{U}_b - \tau \overline{T}_n + k_b \overline{U}_b - \overline{p}_b^{(ex)} \tag{8i}$$

$$\frac{d\overline{M}_t}{ds} = z^2 \rho I_t \overline{\Omega}_t + \chi \overline{M}_n + (k_1)_t \overline{\Omega}_t - \overline{m}_t^{(ex)} \tag{8j}$$

$$\frac{d\overline{M}_n}{ds} = z^2 \rho I_n \overline{\Omega}_n + \tau \overline{M}_b - \chi \overline{M}_t + \overline{T}_b + (k_1)_n \overline{\Omega}_n - \overline{m}_n^{(ex)} \tag{8k}$$

$$\frac{d\overline{M}_b}{ds} = z^2 \rho I_b \overline{\Omega}_b - \tau \overline{M}_n - \overline{T}_n + (k_1)_b \overline{\Omega}_b - \overline{m}_b^{(ex)} \tag{8l}$$

where k and k_1 are the spring constants. χ and τ are parameters that define the geometry of the rod. When $\chi = \tau = 0$, it represents the straight rods and when $\tau = 0$, $\chi = 1/R$ *and* $ds = R\, d\phi$, it represents the circular rods (Fig. 1).

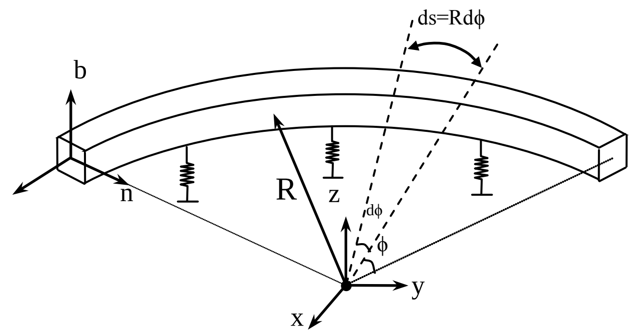

Fig. 1. Circular beam on elastic foundation

For the forced vibration analysis, a column matrix $\mathbf{Y}(s, t)$ is defined as

$$\mathbf{Y}(s,t) = \{U_t^o,\ U_n^o,\ U_b^o,\ \Omega_t^o,\ \Omega_n^o,\ \Omega_b^o,\ T_t^o,\ T_n^o,\ T_b^o,\ M_t^o,\ M_n^o,\ M_b^o\}^T \tag{9}$$

Laplace transform of equation (9) with respect to time $L[\mathbf{Y}(s,t)] = \overline{\mathbf{Y}}(s,z)$, for t>0 is defined as

$$\overline{\mathbf{Y}}(s,z) = \int_0^\infty \mathbf{Y}(s,t)\, e^{-zt}\, dt \tag{10}$$

where the Laplace transform parameter z is a complex number.

For the free vibration analysis, we set $p_i^{(ex)} = 0$ and $m_i^{(ex)} = 0$ with (i= t, n, b). Assuming harmonic motion, $\mathbf{U}^o$, $\mathbf{\Omega}^o$, $\mathbf{T}^o$ and $\mathbf{M}^o$ take the form

$$\mathbf{U}^o(s,t) = \mathbf{U}^*(s) e^{i\omega t},\ \mathbf{\Omega}^o(s,t) = \mathbf{\Omega}^*(s) e^{i\omega t},\ \mathbf{T}^o(s,t) = \mathbf{T}^*(s) e^{i\omega t},\ \mathbf{M}^o(s,t) = \mathbf{M}^*(s) e^{i\omega t} \tag{11}$$

and substituting (11) into (2-3) a set of twelve first-order linear, homogeneous ordinary differential equations is obtained. If the generalized displacements U_t^*, U_n^*, U_b^*, Ω_t^*, Ω_n^*, Ω_b^* and corresponding generalized resultant forces

T_t^*, T_n^*, T_b^*, M_t^*, M_n^*, M_b^* are considered as the components, in the indicated order, of a column matrix $\mathbf{Y}^*(s)$, these twelve equations can be rewritten in the matrix form as

$$\frac{\mathrm{d}\,\mathrm{Y}^*(s)}{\mathrm{d}\,s} = \mathrm{F}(s,\omega)\mathrm{Y}^*(s) \tag{12}$$

The values of ω which make the determinant of the system dynamic stiffness matrix zero are the natural frequencies of the problem. For the case of free vibrations the dynamic stiffness matrix is obtained by applying the complementary functions method described [7-9]. Both the element dynamic stiffness matrix and the fixed-end forces are determined by the method of the complementary functions in the Laplace domain [7-9].

3. Numerical examples

In this section, various problems are presented. First, in order to validate the present model, the free vibration frequencies of straight beam on Winkler-type elastic foundation are compared with the results available in the literature. In addition, this system is also analyzed under an impulsive load. Second, a clamped-clamped circular beam on elastic foundation is considered. The effect of R/h ratio on dynamic behavior is also investigated.

Example 1. A simply supported uniform beam of length L = 6.096 m resting on an elastic foundation is considered (Fig.2). The beam has Young's modulus E = 24.82 GPa, material density ρ = 3387 kg/m^3, ν = 0.3 and second moment of inertia I = 144×10^{-5} m^4. The stiffness of foundation is k = 16.55 MN/m^2. Free vibration frequencies calculated by using the present computer program are given in Table 1. It can be seen from Table 1 that the results of the present model demonstrate good agreement with the previously obtained result and ANSYS predictions.

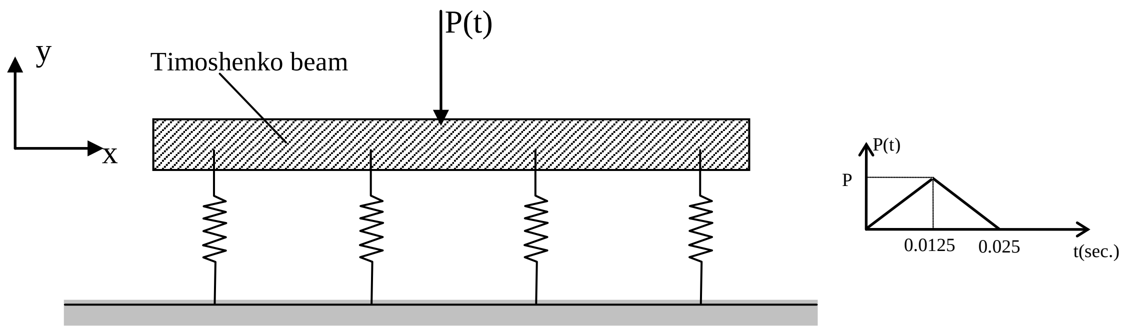

Fig. 2. A Timoshenko beam on elastic foundation and triangular impulsive load.

After having tested the validity of the present model on the free vibration problem, the forced vibration analysis of beam on elastic foundation is presented. A triangular impulsive load (Fig. 2) with the amplitude P = 100 kN is applied at the midpoint of the beam.

Displacement at the midpoint of the beam are compared with the results of the ANSYS (Fig. 3).

Table 1. The natural frequencies for simply supported beam on elastic foundation (Hz)

Mode	Timoshenko et al. [11]	Lai et al. [12]	Thambiratnam and Zhuge [13]	Friswell et al. [14]	ANSYS [15]	Present study
1	32.903	32.905	32.903	32.898	32.862	32.863
2	56.814	56.822	56.819	56.808	56.589	56.597
3	112.91	111.97	111.96	111.90	110.74	110.76

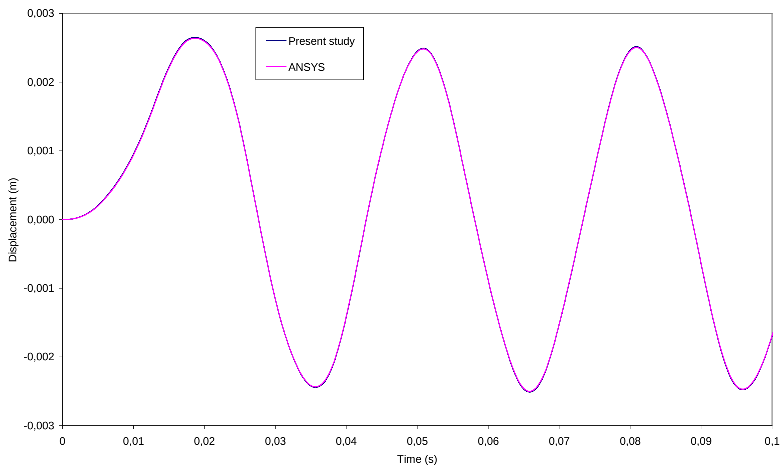

Fig. 3. Displacement versus time at the midpoint of the beam

Example 2. A semicircular beam on elastic foundation fixed at both ends is considered. The material and geometrical properties of the system are $R = 7.63\ m$, $b = h = 0.763$ m, $E = 47.24\ GPa$, $G = 19.68\ GPa$, $k_b = 23.623\ MPa$, $(k_1)_t = 1143\ kNm/m$. This example has also been used in the work of Çalım and Akkurt [8] for analyzing the free vibration of circular beam on elastic foundation. The natural frequencies of circular beam on elastic foundation are calculated and given in Table 2.

A triangular impulsive load with the amplitude P = 100 kN is applied at the midpoint of the beam. Displacement (Uz) at the midpoint of the beam is compared with the results of the ANSYS (Fig. 4). These results obtained using the present method exhibit very good agreement with those obtained using ANSYS.

The effect of the ratio R/h on dynamic behavior of the circular beam on elastic foundation is investigated. The results are compared to those obtained from ANSYS. It is observed that when the ratio of R/h of semicircular beam on elastic foundation increases, the displacement amplitude increases as well while the vibration period decreases.

Table 2. The fundamental natural frequencies semicircular beam fixed at both ends (Hz)

R/h	ANSYS	Present Study		R/h	ANSYS	Present Study	
	$(k_1)_t=0$	$(k_1)_t=0$	$(k_1)_t\neq0$		$(k_1)_t=0$	$(k_1)_t=0$	$(k_1)_t\neq0$
5	9.727	9.730	9.733	**17**	24.432	24.433	24.444
7	11.088	11.090	11.094	**19**	27.258	27.260	27.272
9	13.406	13.408	13.414	**21**	30.168	30.169	30.182
11	16.038	16.039	16.047	**23**	32.972	32.973	32.987
13	18.799	18.800	18.809	**25**	35.882	35.883	35.897
15	21.595	21.596	21.606				

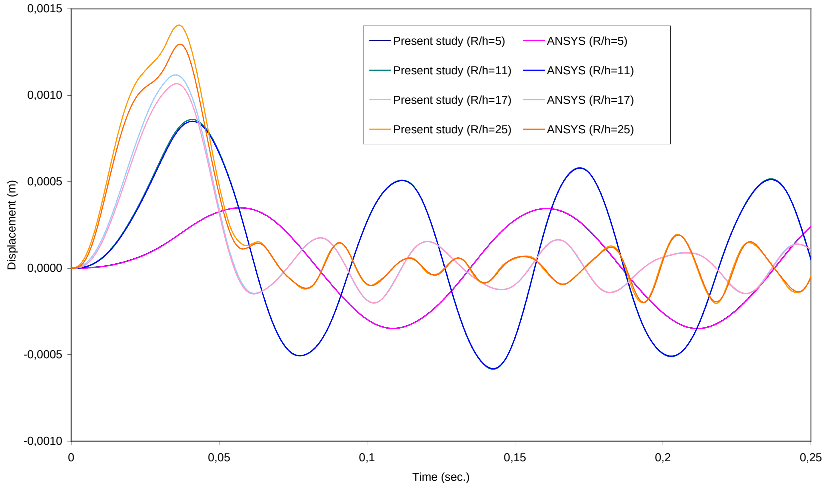

Fig. 4. Uz displacement versus time midpoint of the semicircular beam

4. Conclusions

The free and forced vibration analysis of straight and circular beams on elastic foundation is investigated. A computer program is coded in Fortran to perform the analysis in the Laplace domain. The dynamic stiffness matrix has been calculated in the Laplace domain by applying the complementary functions method to the differential equations in canonical form. Free and forced vibrations calculated to validate the developed computer program are compared with the data given in the literature and ANSYS.

As the ratio R/h of circular beam on elastic foundation increase, the natural frequencies increase. When the ratio, R/h, of circular beam on elastic foundation increases, the displacement amplitude increases as well while the vibration period decreases.

Acknowledgements

This study is supported by the Scientific and Technical Research Council of Turkey (TUBITAK-108M593).

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