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Mode-Shapes Analysis Of Cross-Ply-Laminated Composite Cylindrical Shallow Shells

Ali Dogan

Department of Civil Engineering, Mustafa Kemal University, 31200 Hatay, Turkiye

ABSTRACT

This work presents mode-shape analysis of cross-ply laminated composite cylindrical shallow shells. First, kinematic relations of strains and deformation have been showed. Then, using Hamilton's principle, governing differential equations have been obtained for a general curved shell. In the next step, stress-strain relation for laminated, cross-ply composite shells has been given. By using some simplifications and assuming Fourier series as a displacement field, differential equations are solved by matrix algebra for shallow shells. By the help of a computer algebra system called MATHEMATICA, a computer program has been prepared for the solution. The results obtained by this solution have been given tables and graphs. Example problems have been solved also by (ANSYS and SAP2000) programs, which are based on the finite element method (FEM). The comparison has been made using all of tables and graphs that are obtained by various theories.

Keywords: *Structural composites; Vibration; Anisotropy; Shell theory; Finite Element Method (FEM).*

1. Introduction

A composite is a structural material, which consists of combining two or more constituents on a macroscopic scale to form a useful material. The goal of this three dimensional composition is to obtain a property which none of the constituents possesses. In other words, the target is to produce a material that possesses higher performance properties than its constituent parts for a particular purpose. Some of these properties are mechanical strength, corrosion resistance, high temperature resistance, heat conductibility, stiffness, lightness and appearance. In accordance with this definition, the following conditions must be satisfied by the composite material. It must be manmade and unnatural. It must comprise of at least two different materials with different chemical components separated by distinct interfaces. Different materials must be put together in a three dimensional body. It must possess properties, which none of the constituents possesses alone and that must be the aim of its production. The material must behave as a whole, i.e. the fiber and the matrix material (material surrounding the fibers) must be perfectly bonded. Structures composed of composite materials offer lower weight and higher strength and stiffness than those composed of most metallic materials [1].

Shells are common structural elements in many engineering structures, including concrete roofs, exteriors of rockets, ship hulls, automobile tires, containers of liquids, oil tanks, pipes, aerospace etc. A shell can be defined as a curved, thin-walled structure. It can be made from a single layer or multilayer of isotropic or anisotropic materials. Shells can be classified according to their curvatures. Shallow shells are defined as shells that have rise of not more than one fifth the smallest planform dimension of the shell [1].

Shells are three-dimensional (3D) bodies bounded by two relatively close, curved surfaces. The 3D equations of elasticity are complicated that's why all shell theories (thin, thick, shallow and deep, etc.) reduce the 3D elasticity problem into a 2D one. This is done usually by Classical Lamination Theory-CLT and Kirchhoff hypothesis.

A number of theories exist for layered shells. Many of these theories were developed originally for thin shells and based on the Kirchhoff–Love kinematic hypothesis that straight lines normal to the undeformed mid-surface remain straight and normal to the middle surface after deformation. Among these theories Qatu [1] uses energy functional to develop equation of motion. Many studies have been performed on characteristics of shallow shells [2-6]. Recently, Latifa and Sinha [7] have used an improved finite element model for the bending and free vibration analysis of doubly curved, laminated composite shells having spherical and ellipsoidal shapes. Large-amplitude vibrations of circular cylindrical shells subjected to radial harmonic excitation in the spectral neighborhood of the lowest resonance are investigated by Amabili [8]. Gautham and Ganesan [9] deal with the free vibration characteristics of isotropic and laminated orthotropic spherical caps. Liew et al. [10] has presented the elasticity solutions for free vibration analysis of doubly curved shell panels of rectangular planform. Grigorenko and Yaremchenko [11] have analyzed the stress-strain state of a shallow shell with rectangular planform and varying thickness. Djoudi and Bahai [12] have presented a cylindrical strain based shallow shell finite element which is developed for linear and geometrically non-linear analysis of cylindrical shells.

The aim of this study is to compare the frequency parameters of the each mode obtained by the theories given in literature and obtained by the finite element formulation for different cases. Formulations for thick and thin shell theories, given Qatu [1], have been studied and a computer program has been written using those formulations by Mathematica [20] programing language. The solutions of problem by finite element method have been performed by commercial programs, named ANSYS [14] and SAP2000. Results obtained by different theories have been compared for different plan-form dimensions, lamination thickness, ratio of radius of curvature equals to 0.1 and elasticity ratio equals to 15 cases. The shell, that has been examined, has quadrangle plan-form varying from square to rectangle. Moreover, lamination thickness has been taken variable. For different lamination thicknesses, results of the theories are presented by tables. As a material, cross-ply, 4-layred lamination has been choosen. Elasticty ratio is 15. The results obtained from theories have been compared with literature, ANSYS and SAP2000 by using tables and graphs.

2. Theories

A lamina is made of isotropic homogeneous reinforcing fibers and an isotropic homogeneous material surrounding the fibers, called matrix material (Fig. 1). Therefore, the stiffness of the lamina varies from point to point depending on whether the point is in the fiber, the matrix or the fiber and matrix interface. Because of these variations, macro-mechanical analysis of a lamina is based on average properties.

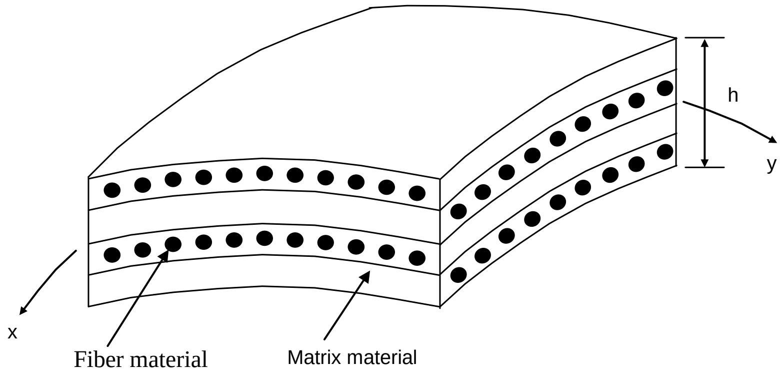


Fig.1. Fiber and matrix materials in laminated composite shallow shell

There are many theories of shells. Classical shell theory, also known as Kirchhoff-Love kinematic hypothesis, assumes that "The normals to the middle surface remain straight and normal to the mid-surface when the shell undergoes deformation". However, according to first order shear deformation theory "The transverse normals do not remain perpendicular to the mid-surface after deformation" [15]. In addition, classical lamination theory says "laminas are perfectly bonded" [16-19]. The theory of shallow shells can be obtained by making the following additional assumptions to thin (or classical) and thick (or shear deformation) shell theories. It will be assumed that the deformation of the shells is completely determined by the displacement of its middle surface. The derivation of equations of motion is based on two assumptions. The first assumption is that the shallow shell has small deflections. The second assumption is that the shallow shell thickness is small compared to its radii of curvature. Also, the radii of curvature are very large compared to the in-plane displacement. Curvature changes caused by the tangential displacement component u and v are very small in a shallow shell, in comparison with changes caused by the normal component w.

2.1. Geometric Properties

The vectorial equation of the undeformed surface could be written by the x and y cartesian coordinates as

$$\vec{r} = \vec{r}(x, y) \tag{1}$$

a small increment in $\vec{r}$ vector is given as

$$\overrightarrow{dr} = \vec{r}_{,x}\, dx + \vec{r}_{,y}\, dy \tag{2}$$

where $\bar{r}_{,x}$ is the small increment in x direction and $\bar{r}_{,y}$ is the small increment in y direction (Fig. 2). The differential length of the shell surface could be found by dot product of $d\bar{r}$ by itself

$$ds^2 = \overrightarrow{dr} \bullet \overrightarrow{dr} = A^2 dx^2 + B^2 dy^2 \quad (3)$$

where A and B are referred as Lame parameters and defined as

$$A^2 = \vec{r},_x \bullet \vec{r},_x \qquad B^2 = \vec{r},_y \bullet \vec{r},_y \quad (4)$$

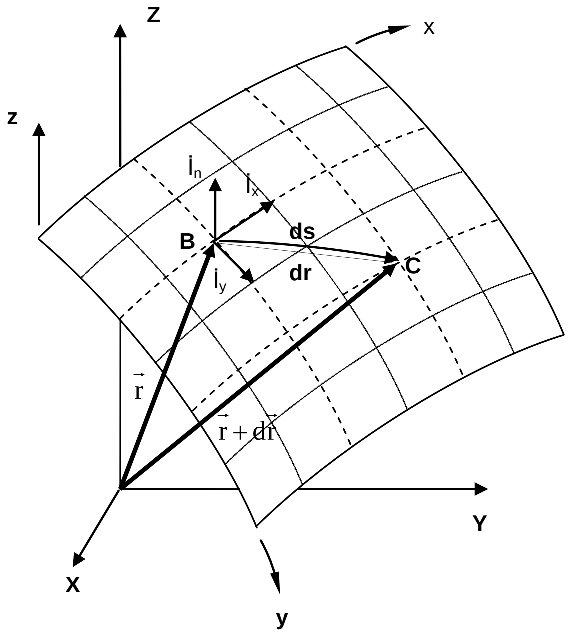


Fig.2. Coordinates of shell mid-surface

Eq. (3) is called first fundamental form of the surface. Tangent vector to the surface could be obtained by taking derivative of Eq. (1) with respect to surface length. Then, applying Frenet's formula to the derivative of tangent vector and multiplying both sides by unit normal vector gives second quadratic form.

2.2. Kinematics of displacement

Let the position of a point, on a middle surface, shown by $\vec{r}(x, y)$. If this point undergoes the displacement by the amount of $\overrightarrow{U}$ then, final position of that point could be given as

$$\vec{r}\,'(x, y) = \vec{r}\,(x, y) + \overrightarrow{U} \quad (5)$$

where $\overrightarrow{U}$ is the displacement field of the point and defined as

$$\overrightarrow{U} = u\vec{i}_x + v\vec{i}_y + w\vec{i}_z \quad (6)$$

where $\vec{i}_x$, $\vec{i}_y$ and $\vec{i}_z$ are the unit vectors in the direction of x, y and z. u, v, and w are the displacements in the direction of x, y and z respectively. Using Eqs. (5) and (6) strains are calculated as

$$
\begin{aligned}
\varepsilon_x &= \frac{1}{(1+z/R_x)}\left(\frac{1}{A}\frac{\partial u}{\partial x}+\frac{v}{AB}\frac{\partial A}{\partial y}+\frac{w}{R_x}\right)\\
\varepsilon_y &= \frac{1}{(1+z/R_y)}\left(\frac{1}{B}\frac{\partial v}{\partial y}+\frac{u}{AB}\frac{\partial B}{\partial x}+\frac{w}{R_y}\right)\\
\varepsilon_z &= \partial w/dz\\
\gamma_{xy} &= \frac{1}{(1+z/R_x)}\left(\frac{1}{A}\frac{\partial v}{\partial x}-\frac{u}{AB}\frac{\partial A}{\partial y}+\frac{w}{R_{xy}}\right)+\frac{1}{(1+z/R_y)}\left(\frac{1}{B}\frac{\partial u}{\partial y}-\frac{v}{AB}\frac{\partial B}{\partial x}+\frac{w}{R_{xy}}\right)\\
\gamma_{xz} &= \frac{1}{A(1+z/R_x)}\frac{\partial w}{\partial x}+A(1+z/R_x)\frac{\partial}{\partial z}\left(\frac{u}{A(1+z/R_x)}\right)-\frac{v}{R_{xy}(1+z/R_x)}\\
\gamma_{yz} &= \frac{1}{B(1+z/R_y)}\frac{\partial w}{\partial y}+B(1+z/R_y)\frac{\partial}{\partial z}\left(\frac{v}{B(1+z/R_y)}\right)-\frac{u}{R_{xy}(1+z/R_y)}
\end{aligned}
\tag{7}
$$

where R_x, R_y, and R_{xy} are curvatures in x-plane, y-plane and xy- plane respectively.

2.3. Stress Strain relation

For an orthotropic media there are 9 stiffness coefficients written in local coordinates.

$$[\sigma] = [Q][\varepsilon] \tag{8}$$

where $[\sigma]$ is the stress matrices, [Q] is the stiffness matrices and $[\varepsilon]$ strain matrices. The stresses in global coordinates are calculated by applying transformation rules. Then, the stresses over the shell thickness are integrated to obtain the force and moment resultants. Due to curvatures of the structure, extra terms must be taken into account during the integration. This difficulty could be overcame by expanding the term $[1/(1+z/R_n)]$ in a geometric series.

2.4. Governing Equations

Equation of motion for shell structures could be obtained by Hamilton's principle

$$\delta\int_{t_1}^{t_2}(T+W-U)dt = 0 \tag{9}$$

where T is the kinetic energy of the structure

$$T = \frac{\rho}{2}\int\left\{\left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial v}{\partial t}\right)^2 + \left(\frac{\partial w}{\partial t}\right)^2\right\}dxdydz \tag{10}$$

W is the work of the external forces

$$W = \int_x\int_y (q_x u + q_y v + q_z w + m_x \psi_x + m_y \psi_y)ABdxdy \tag{11}$$

in which q_x, q_y, q_z are the external forces u, v, w are displacements in x, y, z direction respectively. m_x, m_y, are the external moments and ψ_x, ψ_y are rotations in x, y directions respectively. U is the strain energy defined as,

$$U = \frac{1}{2}\int\left(\sigma_x\varepsilon_x + \sigma_y\varepsilon_y + \sigma_z\varepsilon_z + \sigma_{xy}\varepsilon_{xy} + \sigma_{xz}\varepsilon_{xz} + \sigma_{yz}\varepsilon_{yz}\right)dxdydz \tag{12}$$

Solving Eq. (9) gives set of equations called equations of motion for shell structures.

$$\begin{aligned}
&\frac{\partial}{\partial x}(BN_x) + \frac{\partial}{\partial y}(AN_{yx}) + \frac{\partial A}{\partial y}N_{xy} - \frac{\partial B}{\partial x}N_y + \frac{AB}{R_x}Q_x + \frac{AB}{R_{xy}}Q_y + ABq_x \\
&\qquad = AB(\bar{I}_1\ddot{u}^2 + \bar{I}_2\ddot{\psi}_x^2) \\
&\frac{\partial}{\partial y}(AN_y) + \frac{\partial}{\partial x}(BN_{xy}) + \frac{\partial B}{\partial x}N_{yx} - \frac{\partial A}{\partial y}N_x + \frac{AB}{R_y}Q_y + \frac{AB}{R_{xy}}Q_x + ABq_y \\
&\qquad = AB(\bar{I}_1\ddot{v}^2 + \bar{I}_2\ddot{\psi}_y^2) \\
&-AB\left(\frac{N_x}{R_x} + \frac{N_y}{R_y} + \frac{N_{xy} + N_{yx}}{R_{xy}}\right) + \frac{\partial}{\partial x}(BQ_x) + \frac{\partial}{\partial y}(AQ_y) + ABq_z \\
&\qquad = AB(\bar{I}_1\ddot{w}^2) \\
&\frac{\partial}{\partial x}(BM_x) + \frac{\partial}{\partial y}(AM_{yx}) + \frac{\partial A}{\partial y}M_{xy} - \frac{\partial B}{\partial x}M_y - ABQ_x + \frac{AB}{R_x}P_x + ABm_x \\
&\qquad = AB(\bar{I}_2\ddot{u}^2 + \bar{I}_3\ddot{\psi}_x^2) \\
&\frac{\partial}{\partial y}(AM_y) + \frac{\partial}{\partial x}(BM_{xy}) + \frac{\partial B}{\partial x}M_{yx} - \frac{\partial A}{\partial y}M_x - ABQ_y + \frac{AB}{R_y}P_y + ABm_y \\
&\qquad = AB(\bar{I}_2\ddot{v}^2 + \bar{I}_3\ddot{\psi}_y^2)
\end{aligned} \tag{13}$$

When the shell has small curvature it is referred to as a shallow shell. Shallow shells are defined as shells that have a rise of not more than 1/5th the smallest planform dimension of the shell [1]. It has been widely accepted that shallow shell equations should not be used for maximum span to minimum radius ratio of 0.5 or more. For shallow shells, Lame parameters are assumed to equal to one (A=B=1). This gives Eq. (13) in simplified form as,

$$\frac{\partial}{\partial x}N_x+\frac{\partial}{\partial y}N_{yx}+q_x=\bar{I}_1\ddot{u}^2+\bar{I}_2\ddot{\psi}_x^2 \quad \frac{\partial}{\partial y}N_y+\frac{\partial}{\partial x}N_{xy}+q_y=\bar{I}_1\ddot{v}^2+\bar{I}_2\ddot{\psi}_y^2$$

$$-\left(\frac{N_x}{R_x}+\frac{N_y}{R_y}+\frac{N_{xy}+N_{yx}}{R_{xy}}\right)+\frac{\partial}{\partial x}Q_x+\frac{\partial}{\partial y}Q_y+q_z=\bar{I}_1\ddot{w}^2 \qquad (14)$$

$$\frac{\partial}{\partial x}M_x+\frac{\partial}{\partial y}M_{yx}-Q_x+m_x=\bar{I}_2\ddot{u}^2+\bar{I}_3\ddot{\psi}_x^2$$

$$\frac{\partial}{\partial y}M_y+\frac{\partial}{\partial x}M_{xy}-Q_y+m_y=\bar{I}_2\ddot{v}^2+\bar{I}_3\ddot{\psi}_y^2$$

Eq. (14) is defined as equation of motion for thick shallow shell. The force and moment resultants are

$$\begin{bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{bmatrix}=\begin{bmatrix} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} \\ B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} \end{bmatrix}\begin{bmatrix} \varepsilon_{0x} \\ \varepsilon_{0y} \\ \gamma_{0xy} \\ \kappa_x \\ \kappa_y \\ \tau \end{bmatrix} \qquad (15)$$

$$\begin{bmatrix} Q_x \\ Q_y \end{bmatrix}=\begin{bmatrix} A_{55} & A_{45} \\ A_{45} & A_{44} \end{bmatrix}\begin{bmatrix} \gamma_{0xz} \\ \gamma_{0yz} \end{bmatrix}$$

The Navier type solution can be applied to thick and thin shallow shells. This type solution assumes that the displacement field of the shallow shells could be represented as sine and cosine trigonometric functions.

Consider a shell with shear diaphragm boundaries on all four edges. That is, boundary conditions for simply supported thick shells,

$$\begin{aligned} &N_x=w_0=v_0=M_x=\psi_y=0 && x=0,a \\ &N_y=w_0=u_0=M_y=\psi_x=0 && y=0,b \end{aligned} \qquad (16)$$

The displacement functions of satisfied the boundary conditions apply;

$$u_0(x,y,t) = \sum_{m=0}^{M}\sum_{n=0}^{N} U_{mn} Cos(x_m x) Sin(y_n y) Sin(\omega_{mn} t)$$

$$v_0(x,y,t) = \sum_{m=0}^{M}\sum_{n=0}^{N} V_{mn} Sin(x_m x) Cos(y_n y) Sin(\omega_{mn} t)$$

$$w_0(x,y,t) = \sum_{m=0}^{M}\sum_{n=0}^{N} W_{mn} Sin(x_m x) Cos(y_n y) Sin(\omega_{mn} t) \tag{17}$$

$$\psi_x(x,y,t) = \sum_{m=0}^{M}\sum_{n=0}^{N} \psi_{xmn} Cos(x_m x) Sin(y_n y) Sin(\omega_{mn} t)$$

$$\psi_y(x,y,t) = \sum_{m=0}^{M}\sum_{n=0}^{N} \psi_{ymn} S\mathit{in}(x_m x) C\mathit{os}(y_n y) Sin(\omega_{mn} t)$$

Where, $\alpha_m = \frac{m\pi}{a}, \quad \beta_n = \frac{n\pi}{b}, \quad \omega_{mn}$ is national frequency.

Where, $U_{mn}, V_{mn}, W_{mn}, \psi_{\alpha mn}, \psi_{\beta mn}$ are arbitrary coefficients.

Substituting the above equations into equation of motion and using a Fourier expansion.

$$\begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} & K_{15} \\ K_{21} & K_{22} & K_{23} & K_{24} & K_{25} \\ K_{31} & K_{32} & K_{33} & K_{34} & K_{35} \\ K_{41} & K_{42} & K_{43} & K_{44} & K_{45} \\ K_{51} & K_{52} & K_{53} & K_{54} & K_{55} \end{bmatrix} \begin{bmatrix} U_{mn} \\ V_{mn} \\ W_{mn} \\ \psi_{xmn} \\ \psi_{ymn} \end{bmatrix}$$

$$+\omega_{mn}^2 \begin{bmatrix} -I_1 & 0 & 0 & -I_2 & 0 \\ 0 & -I_1 & 0 & 0 & -I_2 \\ 0 & 0 & -I_1 & 0 & 0 \\ -I_2 & 0 & 0 & -I_3 & 0 \\ 0 & K_{52} & 0 & 0 & -I_3 \end{bmatrix} \begin{bmatrix} U_{mn} \\ V_{mn} \\ W_{mn} \\ \psi_{xmn} \\ \psi_{ymn} \end{bmatrix} = \begin{bmatrix} -P_x \\ -P_y \\ P_n \\ m_x \\ m_y \end{bmatrix} \tag{18}$$

Following equation can be used directly to find the natural frequencies of free vibrations.

$$[K]\{\Delta\} + (\omega_{mn})^2 [M]\{\Delta\} = 0 \tag{19}$$

$$K_{12} = K_{21} = -(A_{12} + A_{66})x_m y_n$$

$$K_{11} = -A_{11}x_m^2 - A_{66}\,y_n^2$$

$$K_{13} = K_{31} = \left[\frac{A_{11}}{R_x} + \frac{A_{12}}{R_y}\right]x_m$$

$$K_{14} = K_{41} = -B_{11}x_m^2 - B_{66}\,y_n^2$$

$$K_{15} = K_{51} = -(B_{12} + B_{66})x_m y_n$$

$$K_{22} = -A_{66}x_m^2 - A_{22}\,y_n^2$$

$$K_{23} = K_{32} = \left[\frac{A_{12}}{R_x} + \frac{A_{22}}{R_y}\right]y_n$$

$$K_{24} = K_{42} = -(B_{12} + B_{66})x_m y_n$$

$$K_{22} = -B_{66}x_m^2 - B_{22}\,y_n^2$$

$$K_{33} = -A_{55}\,x_m^2 - A_{44}\,y_n^2 - \left[\frac{A_{11}}{R_x^2} + \frac{2A_{12}}{R_x R_y} + \frac{A_{22}}{R_y^2}\right] \qquad (20)$$

$$K_{34} = K_{43} = \left[-A_{55} + \frac{B_{11}}{R_x} + \frac{B_{12}}{R_y}\right]x_m$$

$$K_{35} = K_{53} = \left[-A_{44} + \frac{B_{12}}{R_x} + \frac{B_{22}}{R_y}\right]y_n$$

$$K_{44} = -A_{55} - D_{11}\,x_m^2 - D_{66}\,y_n^2$$

$$K_{45} = K_{54} = -(D_{12} + D_{66})x_m y_n$$

$$K_{55} = -A_{44} - D_{66}\,x_m^2 - D_{22}\,y_n^2$$

$$M_{ij} = M_{ji}$$

$$M_{11} = M_{22} = M_{33} = -I_1 \qquad M_{14} = M_{25} = -I_2 \qquad M_{44} = M_{55} = -I_3$$

all other $M_{ij} = 0$

3. Numerical Examples and Conclusions

As an example, a simply supported cylindrical shell which has a ratio of radius of curvature (ratio of shell width/shell radius) equals to 0.1 in one plane and infinite radius of curvature in other plane, has been considered (Fig. 3). The shell, in hand, has a quadrangle planform where the ratio of plan-form dimensions varies from 1 to 4 (a/b=1, 2, 4). As a material, a laminated composite has been used with a [0°/90°/90°/0°] symmetrical cross-ply stacking sequence. Ratio of modulus of elasticity (E_1/E_2) which is the ratio of modulus of elasticity in fiber direction to matrix direction, has been taken as 15. Effect of shell thickness ratio that ratio of shell width to shell thickness, a/h=100 (thick shell) and a/h=10 (thin shell) has been examined.

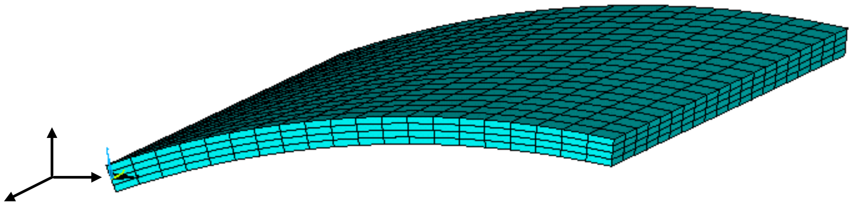


Fig.3. Cylindrical shallow shell

For each case, the shell has been solved with two theories. First theory used in the solution of composite laminated shallow shell is shear deformation shallow shell theory (SDSST). The second theory is the Finite element model (FEM). Entire structure is meshed by finite elements in this theory. Then assuming a suitable displacement fields for each meshing element, the behavior of the structure has been obtained. In this paper, two different finite element package programs, ANSYS and SAP2000 have been used. The structure is meshed by 25×25 elements in ANSYS model. A 8-noded quadratic element is considered as a meshing element named as SHELL99 [14]. The element has 100 layers to model the composite materials used in the structure. For each layer geometric and material properties is entered to program. Furthermore, thicknesses of each layer, fiber orientations and stacking sequence must be entered carefully. During solution process, subspace and block Lanczos mode extracting methods used separately to calculate first 30 frequencies. SAP2000 structural analysis packet program has been used also to verify ANSYS results. Four-node quadrilateral shell elements have been used by SAP2000 finite element program. Orthotropic and lamination properties of the problem could be modeled by using this element.

The governing Eq. (14) (using SDSST theory) derived in the theory section are solved by using Mathematica program separately. Furthermore, ANSYS packet program has been used in solution. The geometry of the shell structures has been created using arc-length method in ANSYS. Then, area element has been defined between the arc lines. Finally using SHELL99 finite element, the area has been meshed. Similar procedures have been applied in SAP2000 program. Due to software differences, modeling steps differs from ANSYS to SAP2000. Furthermore, finite elements also differs in both program hence, FEM results have little difference.

The problem defined at the beginning has been solved by FEM and Mathematica program (Fig. 3). The results obtained by FEM and Mathematica, have been compared in tables and graphs.

Table 1. Comparison of the frequencies (ω) and nondimensional frequency parameters ($\Omega = \omega a^2 \sqrt{\rho / E_2 h^2}$) of the shallow shell obtained by Shear Deformation Shell Theory (SDSST) and Finite Element Method (ANSYS & SAP2000) for six modes ($\underline{a/b = 1}, \underline{a/h = 100}, E_1/E_2 = 15, G_{12}/E_2 = G_{13}/E_2 = G_{13}/E_2 = 0.5 \; \upsilon_{12} = 0.25$ and $K^2 = 5/6$)

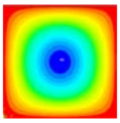
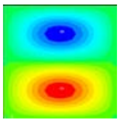
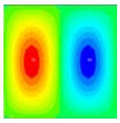
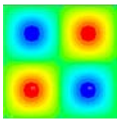
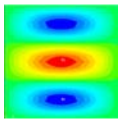
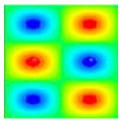
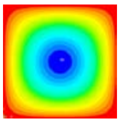
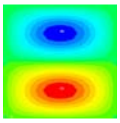
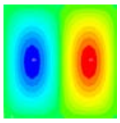
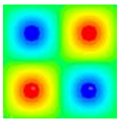
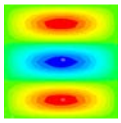
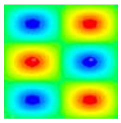
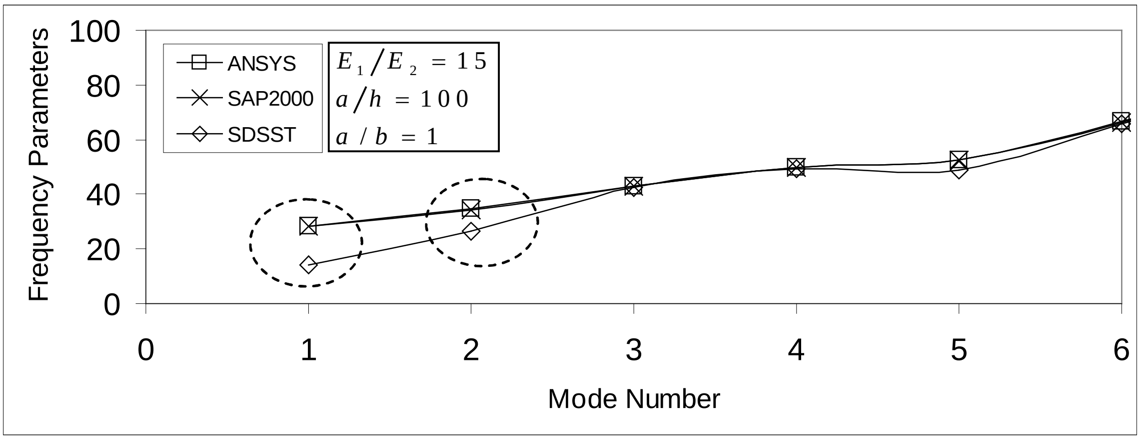
Method	Frequency(ω) and Nondimensional Frequency(Ω) Parameters	Mode Shapes (1,1)	(1,2)	(2,1)	(2,2)	(1,3)	(2,3)
							
ANSYS	Frequency (ω)	0,06355	0,07793	0,09666	0,11253	0,11819	0,14997
	$\Omega = \omega a^2 \sqrt{\rho / E_2 h^2}$	**28,23443**	**34,62393**	**42,94346**	**49,99755**	**52,50920**	**66,62799**
Sap2000	Frequency (ω)	0,06335	0,07753	0,09678	0,11208	0,11811	0,14899
	$\Omega = \omega a^2 \sqrt{\rho / E_2 h^2}$	**28,14700**	**34,44700**	**42,99911**	**49,79672**	**52,47578**	**66,19629**
SDSST	Frequency (ω)	0,03141	0,05973	0,09606	0,11098	0,10982	0,14806
	$\Omega = \omega a^2 \sqrt{\rho / E_2 h^2}$	**13,95614**	**26,53820**	**42,67974**	**49,30593**	**48,79160**	**65,78098**

Table 2. Comparison of the frequencies (ω) and nondimensional frequency parameters ($\Omega = \omega a^2 \sqrt{\rho / E_2 h^2}$) of the shallow shell obtained by Shear Deformation Shell Theory (SDSST) and Finite Element Method (ANSYS & SAP2000) for six modes ($\underline{a/b = 1}, \underline{a/h = 10}, E_1/E_2 = 15, G_{12}/E_2 = G_{13}/E_2 = G_{13}/E_2 = 0.5 \; \upsilon_{12} = 0.25$ and $K^2 = 5/6$)

Method	Frequency(ω) and Nondimensional Frequency(Ω) Parameters	Mode Shapes (1,1)	(1,2)	(2,1)	(2,2)	(1,3)	(2,3)
							
ANSYS	Frequency (ω)	0,25243	0,46610	0,67387	0,79500	0,83811	1,06904
	$\Omega = \omega a^2 \sqrt{\rho / E_2 h^2}$	**11,21531**	**20,70810**	**29,93939**	**35,32112**	**37,23615**	**47,49629**
Sap2000	Frequency (ω)	0,25383	0,46776	0,67790	0,80235	0,83943	1,06658
	$\Omega = \omega a^2 \sqrt{\rho / E_2 h^2}$	**11,27755**	**20,78216**	**30,11844**	**35,64765**	**37,29503**	**47,38708**
SDSST	Frequency (ω)	0,24729	0,46168	0,67160	0,79039	0,82572	1,05663
	$\Omega = \omega a^2 \sqrt{\rho / E_2 h^2}$	**10,98675**	**20,51179**	**29,83849**	**35,11589**	**36,68563**	**46,94494**



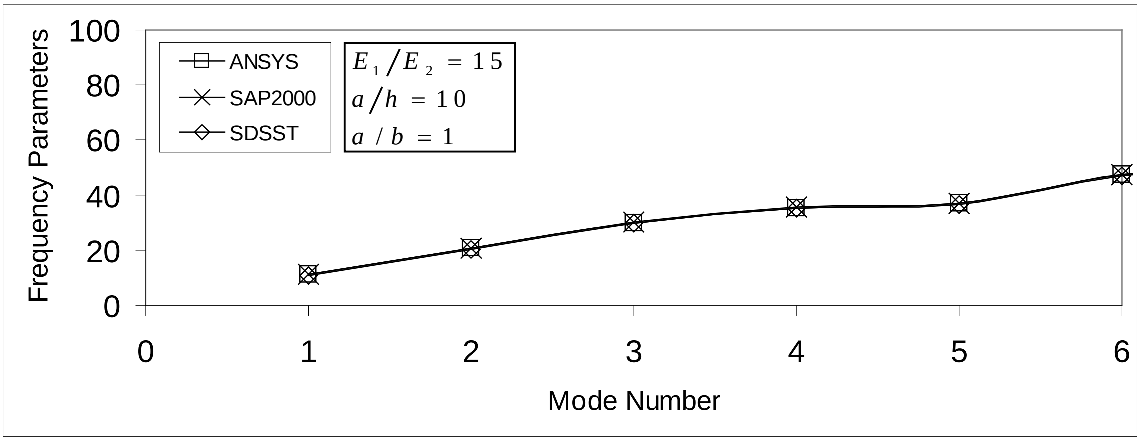


Fig. 4. Comparison of the frequency parameters of the first six modes obtained by SDSST and FEM (For Table 1-2)

Table 3. Comparison of the frequencies (ω) and nondimensional frequency parameters ($\Omega = \omega a^2 \sqrt{\rho / E_2 h^2}$) of the shallow shell obtained by Shear Deformation Shell Theory (SDSST) and Finite Element Method (ANSYS & SAP2000) for six modes ($a/b = 2$, $a/h = 100$, $E_1/E_2 = 15$, $G_{12}/E_2 = G_{13}/E_2 = G_{13}/E_2 = 0.5$ $\upsilon_{12} = 0.25$ and $K^2 = 5/6$)

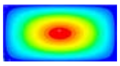
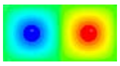
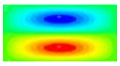
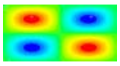
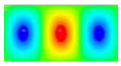
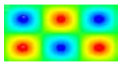
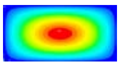
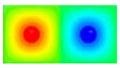
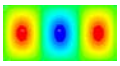
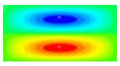
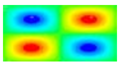
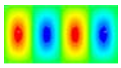
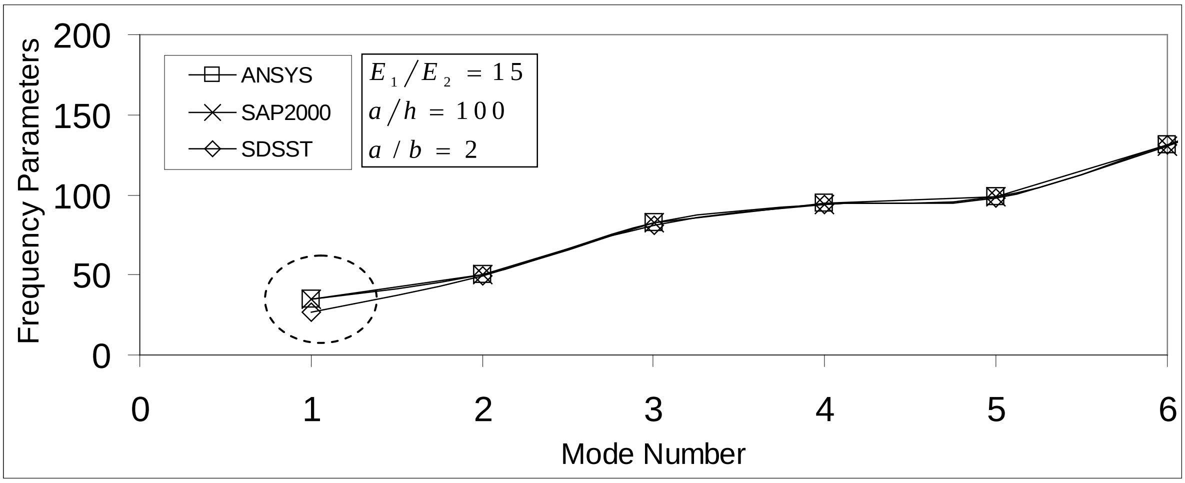
Method	Frequency(ω) and Nondimensional Frequency(Ω) Parameters	Mode Shapes					
		(1,1)	(2,1)	(1,2)	(2,2)	(3,1)	(3,2)
							
ANSYS	Frequency (ω)	0,07792	0,11249	0,18540	0,21282	0,22168	0,29480
	$\Omega = \omega a^2 \sqrt{\rho / E_2 h^2}$	**34,61891**	**49,97761**	**82,36936**	**94,55278**	**98,49128**	**130,97482**
Sap2000	Frequency (ω)	0,07778	0,11259	0,18557	0,21215	0,22282	0,29343
	$\Omega = \omega a^2 \sqrt{\rho / E_2 h^2}$	**34,55585**	**50,02286**	**82,44702**	**94,25532**	**98,99809**	**130,36618**
SDSST	Frequency (ω)	0,05973	0,11098	0,18157	0,21131	0,22081	0,29434
	$\Omega = \omega a^2 \sqrt{\rho / E_2 h^2}$	**26,53820**	**49,30593**	**80,66823**	**93,88395**	**98,10270**	**130,77349**

Table 4. Comparison of the frequencies (ω) and nondimensional frequency parameters ($\Omega = \omega a^2 \sqrt{\rho / E_2 h^2}$) of the shallow shell obtained by Shear Deformation Shell Theory (SDSST) and Finite Element Method (ANSYS & SAP2000) for six modes ($a/b = 2$, $a/h = 10$, $E_1/E_2 = 15$, $G_{12}/E_2 = G_{13}/E_2 = G_{13}/E_2 = 0.5$ $\upsilon_{12} = 0.25$ and $K^2 = 5/6$)

Method	Frequency(ω) and Nondimensional Frequency(Ω) Parameters	Mode Shapes					
		(1,1)	(2,1)	(3,1)	(1,2)	(2,2)	(4,1)
							
ANSYS	Frequency (ω)	0,46414	0,79268	1,24490	1,28418	1,45120	1,71915
	$\Omega = \omega a^2 \sqrt{\rho / E_2 h^2}$	**20,62120**	**35,21768**	**55,30951**	**57,05468**	**64,47528**	**76,37966**
Sap2000	Frequency (ω)	0,46710	0,80322	1,26224	1,28411	1,44645	1,73019
	$\Omega = \omega a^2 \sqrt{\rho / E_2 h^2}$	**20,75257**	**35,68595**	**56,07962**	**57,05168**	**64,26426**	**76,87014**
SDSST	Frequency (ω)	0,46168	0,79039	1,23691	1,26276	1,43127	1,70701
	$\Omega = \omega a^2 \sqrt{\rho / E_2 h^2}$	**20,51179**	**35,11589**	**54,95428**	**56,10285**	**63,58975**	**75,84034**



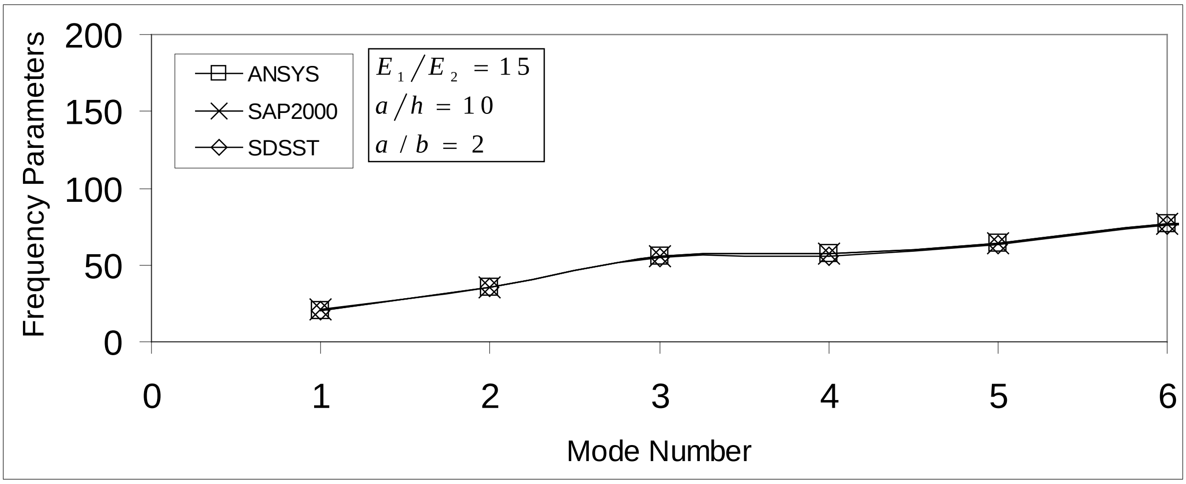


Fig. 5. Comparison of the frequency parameters of the first six modes obtained by SDSST and FEM (For Table 3-4)

Table 5. Comparison of the frequencies (ω) and nondimensional frequency parameters ($\Omega = \omega a^2 \sqrt{\rho / E_2 h^2}$) of the shallow shell obtained by Shear Deformation Shell Theory (SDSST) and Finite Element Method (ANSYS & SAP2000) for six modes ($a/b = 4$, $a/h = 100$, $E_1/E_2 = 15$, $G_{12}/E_2 = G_{13}/E_2 = G_{13}/E_2 = 0.5$ $\upsilon_{12} = 0.25$ and $K^2 = 5/6$)

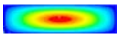
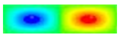
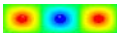
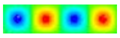
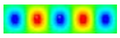
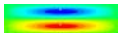
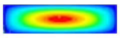
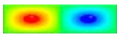
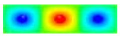
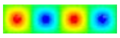
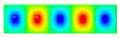
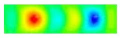
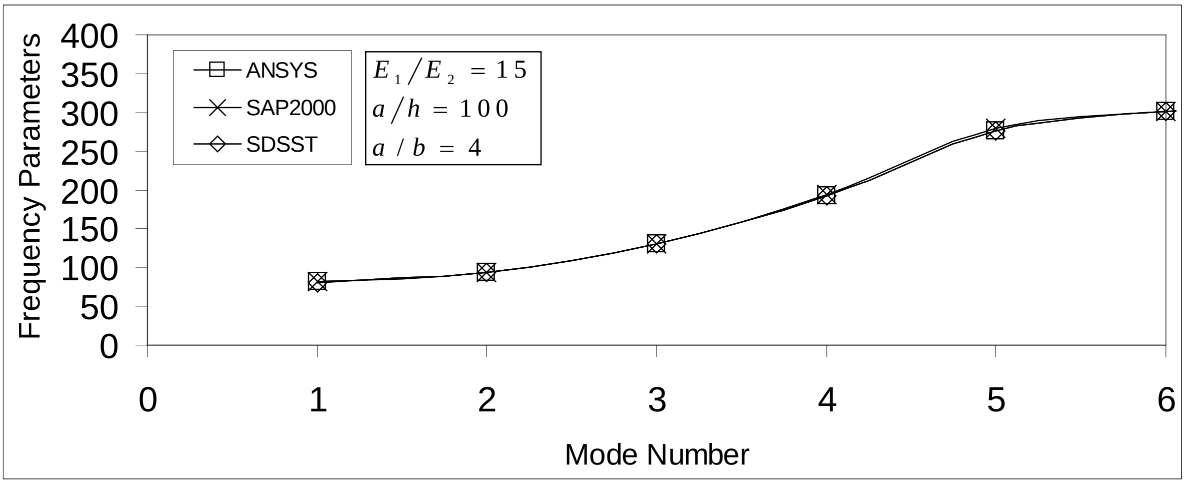
Method	Frequency(ω) and Nondimensional Frequency(Ω) Parameters	Mode Shapes					
		(1,1)	(2,1)	(3,1)	(4,1)	(5,1)	(1,2)
							
ANSYS	Frequency (ω)	0,18532	0,21253	0,29433	0,43302	0,62177	0,67799
	$\Omega = \omega a^2 \sqrt{\rho / E_2 h^2}$	**82,33340**	**94,42237**	**130,76809**	**192,38685**	**276,24350**	**301,22453**
Sap2000	Frequency (ω)	0,18521	0,21197	0,29404	0,43532	0,63037	0,67924
	$\Omega = \omega a^2 \sqrt{\rho / E_2 h^2}$	**82,28619**	**94,17535**	**130,63942**	**193,40802**	**280,06512**	**301,77794**
SDSST	Frequency (ω)	0,18157	0,21131	0,29434	0,43355	0,62192	0,67621
	$\Omega = \omega a^2 \sqrt{\rho / E_2 h^2}$	**80,66823**	**93,88395**	**130,77349**	**192,61986**	**276,31394**	**300,43386**

Table 6. Comparison of the frequencies (ω) and nondimensional frequency parameters ($\Omega = \omega a^2 \sqrt{\rho / E_2 h^2}$) of the shallow shell obtained by Shear Deformation Shell Theory (SDSST) and Finite Element Method (ANSYS & SAP2000) for six modes ($a/b = 4$, $a/h = 10$, $E_1/E_2 = 15$, $G_{12}/E_2 = G_{13}/E_2 = G_{13}/E_2 = 0.5$ $\upsilon_{12} = 0.25$ and $K^2 = 5/6$)

Method	Frequency(ω) and Nondimensional Frequency(Ω) Parameters	Mode Shapes					
		(1,1)	(2,1)	(3,1)	(4,1)	(5,1)	(,)
							
ANSYS	Frequency (ω)	1,28088	1,44541	1,74356	2,11521	2,52312	2,83047
	$\Omega = \omega a^2 \sqrt{\rho / E_2 h^2}$	**56,90792**	**64,21800**	**77,46421**	**93,97609**	**112,09918**	**125,75464**
Sap2000	Frequency (ω)	1,28100	1,44683	1,74748	2,11563	2,50501	2,82076
	$\Omega = \omega a^2 \sqrt{\rho / E_2 h^2}$	**56,91329**	**64,28101**	**77,63831**	**93,99505**	**111,29466**	**125,32320**
SDSST	Frequency (ω)	1,26276	1,43127	1,72622	2,09284	2,49546	
	$\Omega = \omega a^2 \sqrt{\rho / E_2 h^2}$	**56,10285**	**63,58975**	**76,69392**	**92,98233**	**110,87050**	



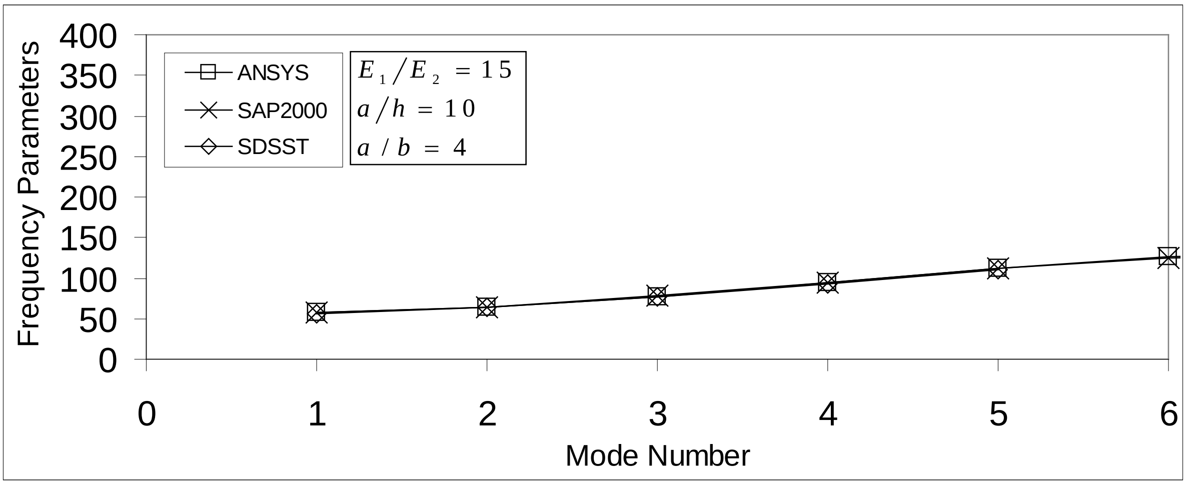


Fig. 6. Comparison of the frequency parameters of the first six modes obtained by SDSST and FEM (For Table 5-6)

A comparison has been made between the SDSST and FEM analysis results. In literature, there are no enough FEM and SDSST results considering shell plan-form dimensions and giving higher modes. Generally, first or first three mode results have been found in literature considering a typical shell plan-form dimension and/or a typical shell thickness. To get knowledge about the agreement of the FEM and SDSST higher modes must be calculated.

In this study, first six non dimensional frequency parameters, obtained from SDSST and FEM, have been given. A few differences have been observed in first mode results of the both theories (Fig. 4 and 5). This differences get smaller and almost zero in higher modes. Hence, observing only first modes using one method could result in misunderstanding the behavior of the structure. All modes must be calculated as a analysis result using both methods.

In the case of thin and square plan-form dimensions of shallow shell, SDSST's and FEM's first mode results differs from each other (Fig.4) This difference gets smaller for second mode and almost zero for higher modes. However, as the shell thickness increases, the results almost coincide for all modes.

The effect of shell plan-form dimensions also has been studied for three cases (a/b=1,2,4). As the shell plan-form changes from square (a/b=1) to rectangle (a/b=4), the results of SDSST and FEM get closer. The difference between first mode results of SDSST and FEM for the case of square, thin shallow shell has been disappeared for the case of rectangle thin shallow shell.

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