International Balkans Conference on Challenges of Civil Engineering, BCCCE,
19-21 May 2011, EPOKA University, Tirana, ALBANIA.

Nonlinear Numerical Model of Thin Plates

Vlaho Akmadzic[1], Ante Mihanovic[2], Boris Trogrlic[3]

[1]*Department of Civil Engineering, University of Mostar, Mostar, B&H*
[2]*Department of Civil Engineering, University of Split, Split, Croatia*
[3]*Department of Civil Engineering, University of Split, Split, Croatia*

ABSTRACT

The article deals with nonlinear analysis of the thin plates. Nonlinear numerical shell model, whose flexural state is based on the Germain-Kirchhoff theory, takes into account geometrical nonlinearity. Material model of the flat shell finite element is isotropic elastic material. The material is defined by the Poisson's coefficient υ and Young modulus of elasticity E.

The geometrical nonlinear numerical model of the space surface structures is presented in short. The numerical model is implemented in the software made in the program language Fortran. As the result of the calculation program gives the space equilibrium path of the desired node, shows the state of the stress field of the surface structure and gives the deformation of the whole structure in each incremental step.

The model application on the surface elements is illustrated by the thin square plate structure. The plate is discretised with the surface finite element. The minimal acceptable finite element discretisation is taken into account. More density discretisation gives more accurate results.

The results, made by commercial software or by another authors or by the theory, are compared with this one.

INTRODUCTION

The space surface structures are analyzed with the numerical model based on the finite element technique. The structure is discretised on the small pieces, finite elements. In that way the complex structure is maximum simplified, because the solution on the first element is repeated on the next one. This fact is very useful for the computer programming and software developing.

The aim of the simulation of the real structure by the numerical model is to find out the kind of behavior of the structure. In other words, the model has to show the deformations and the inner forces in each element of the structure and the stability of the whole structure too.

The program modulus LJUSKA, developed at the Faculty of Civil Engineering University of Mostar, analyses the stability of the space surface structures.

Only the necessary part of the theory, which led to the formulation of the new shell finite element, will be shown. That part is connected with the flexion state of the shell finite element, based on the statement explained at [2].

GEOMETRICAL NONLINEAR MODEL

In reality the equilibrium state of the loaded structure happens in the state of the deformed structure. The consequences are the differences of the equilibrium equations between two states, the undeformed and the deformed. The equations become nonlinear, and harder for solving.

The rigid structures are characterized with small deflection, while the more deformable structures might be characterized with large deflections. The assumption of the theory of stability is equilibrium in the deformed state. Depending on the way of forces showing there are two formulations

- Total Lagrange formulation, and
- Update Lagrange formulation.

This numerical model, based on the Total Lagrange formulation, is shown in [1].

MODEL DEVELOPED ACCORDING TO THE THEORY OF THIN PLATES

The plates are the surface structures which are, depending of the theory of analyses, divided in to two categories, thin and thick plates. Here, we are dealing with the thin plates. The theory of small deformation is used if the maximum deflection of the plate is smaller than the thickness of the plate. If the maximum defection is close to the thickness of the plate the theory of large deflections has to be used.

The thin plate theory is used according to the Germain-Kirchhoff assumptions. The consequences of the assumptions give the vector of relative deformation of the plate in the form

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{xy} \end{bmatrix} = -z \cdot \begin{bmatrix} \dfrac{\partial^2}{\partial x^2} \\ \dfrac{\partial^2}{\partial y^2} \\ \dfrac{\partial^2}{\partial x \partial y} \end{bmatrix} \cdot w \tag{1}$$

Adequate stress vector is

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = -\frac{E \cdot z}{1-\nu^2} \cdot \begin{bmatrix} \dfrac{\partial^2 w}{\partial x^2} + \nu \cdot \dfrac{\partial^2 w}{\partial y^2} \\ \dfrac{\partial^2 w}{\partial y^2} + \nu \cdot \dfrac{\partial^2 w}{\partial x^2} \\ \dfrac{\partial^2 w}{\partial x \partial y} \end{bmatrix} \tag{2}$$

where E represents the Young modulus, and ν represents the Poisson coefficient. The equilibrium at the differential element of the plate is shown at Figure 1.

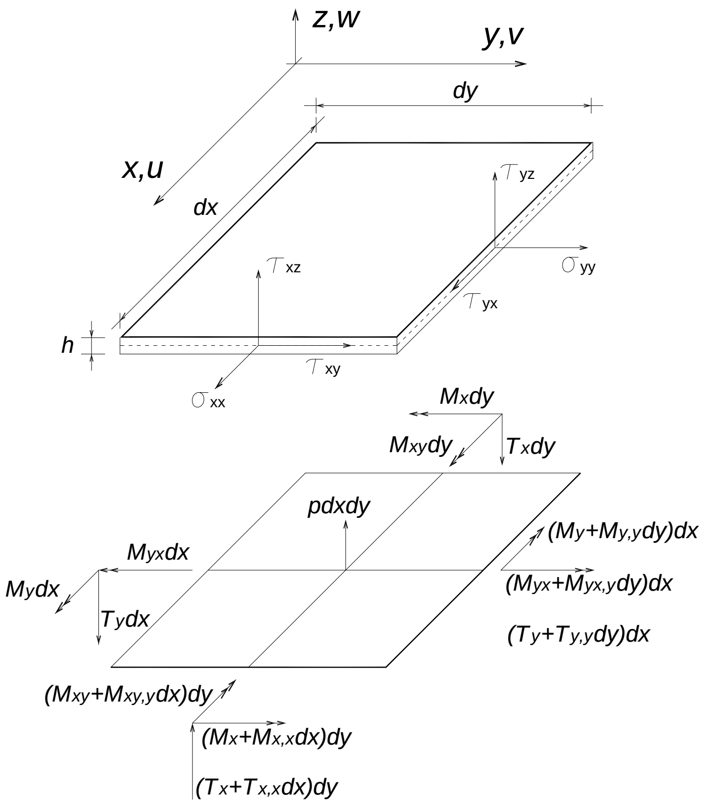

Figure 1 Differential plate element - load and forces

The equilibrium conditions of the thin plate differential element give three partial differential first order equations.

$$\begin{aligned}
&\frac{\partial T_x}{\partial x} + \frac{\partial T_y}{\partial y} + p = 0 \\
&\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} - T_x = 0 \\
&\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} - T_y = 0
\end{aligned} \tag{3}$$

If T forces are eliminated from the equitation (3), the equilibrium conditions could be represented with the differential second order equation.

$$\frac{\partial^2 M_x}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} + p = 0 \tag{4}$$

If the flexural constant of the plate is defined with D_S like

$$D_S = \frac{E \cdot h^3}{12 \cdot (1 - \nu^2)} \tag{5}$$

then the value of each moment from the equation (4) using equation (2) will be

$$M_x = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{xx} \cdot z\, dz = -D_S \cdot \left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} \right)$$

$$M_y = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{yy} \cdot z\, dz = -D_S \cdot \left(\frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2} \right) \qquad (6)$$

$$M_{xy} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{xy} \cdot z\, dz = -(1-\nu) \cdot D_S \frac{\partial^2 w}{\partial x \partial y}$$

If the equations (4), (5) and (6) are connected together and reordered they will provide the new partial forth order differential equation. It describes the deformation form of the plate.

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{p}{D_S} \qquad (7)$$

STATE OF FLEXION ACCORDING TO THE DISPLACEMENT METHOD

If the deflections w of the plate are known in each point of the plate, the displacement field p is defined and could be represented as

$$p \cong \{w(\xi, \eta)\} = \mathbf{H} \cdot \mathbf{u} \qquad (8)$$

where **H** is matrix of the shape functions and **u** is the vector of the displacements. The plate finite element has independent fields of the translational and rotational displacements. For the thin plate element the vector of the unknown displacement $\mathbf{u}_i$ for the each node of the finite element looks like

$$\mathbf{u}_i = \begin{bmatrix} w_i \\ \varphi_i \\ \theta_i \end{bmatrix} \qquad (9)$$

and the corresponding vector of the shape functions $\mathbf{H}_i$ is

$$\mathbf{H}_i = \begin{bmatrix} h_{i1} & h_{i2} & h_{i3} \end{bmatrix} \qquad (10)$$

where h_{i1} represents the displacement shape function, h_{i2} the shape function of the rotation around the axes ξ and h_{i3} the shape function of the rotation around the axes η. The shape functions are taken from the [2].

STATE OF FLEXION – VIRTUAL WORK PRINCIPE

Let the finite element be exposed to the influence of the static forces f. Then, using the virtual work principle, could be written

$$\int_{\Gamma} \mathbf{p}^T \mathbf{s}\, d\delta\Gamma + \int_{\Omega} \delta^{\;T}\; \mathbf{p} d\mathbf{f}\Omega - \int_{\Omega} \delta^{\;T}\; d\Omega = 0 \tag{11}$$

If the equation for the node forces **s** looks like (12), than the node forces vector, in the node *i*, looks like equation (13).

$$\mathbf{s} = \left[\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3, \mathbf{s}_4\right]^T \tag{12}$$

$$\mathbf{s}_i = \left[T_{iz}, M_{ix}, M_{iy}\right]^T \tag{13}$$

Than the vector of the relative deformation **ε** in some point of the finite element could be written as

$$\boldsymbol{\varepsilon} = \mathbf{L}\ \mathbf{p} = \mathbf{L}\ \mathbf{H}\ \mathbf{u} = \mathbf{B}\ \mathbf{u} \tag{14}$$

The above mentioned deformation matrix identify with **B**

$$\mathbf{B} = \left[\mathbf{B}_1, \mathbf{B}_2, \mathbf{B}_3, \mathbf{B}_4\right] \tag{15}$$

is consisting from *n* sub matrix $\mathbf{B}_i$ (i=1,...,n).

$$\mathbf{B}_i = \begin{bmatrix} -\dfrac{\partial^2 h_{i1}}{\partial x^2} & -\dfrac{\partial^2 h_{i2}}{\partial x^2} & -\dfrac{\partial^2 h_{i3}}{\partial x^2} \\ -\dfrac{\partial^2 h_{i1}}{\partial y^2} & -\dfrac{\partial^2 h_{i2}}{\partial y^2} & -\dfrac{\partial^2 h_{i3}}{\partial y^2} \\ -2\dfrac{\partial^2 h_{i1}}{\partial x \partial y} & -2\dfrac{\partial^2 h_{i2}}{\partial x \partial y} & -2\dfrac{\partial^2 h_{i3}}{\partial x \partial y} \end{bmatrix} \tag{16}$$

Stress and strain are connected by the constitutive law, which for the elastic material represents the generalization of the Hooke's law. If the vector of the inner forces looks like

$$\boldsymbol{\sigma} = \left[M_x, M_y, M_{xy}\right]^T \tag{17}$$

than the elastic matrix D for the flexural state of thin plates is described by equation (18).

$$\mathbf{D} = \frac{E \cdot h^3}{12 \cdot (1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \tag{18}$$

Assumption of positive orientations of the inner forces, deflections and deformations is shown at Figure 2.

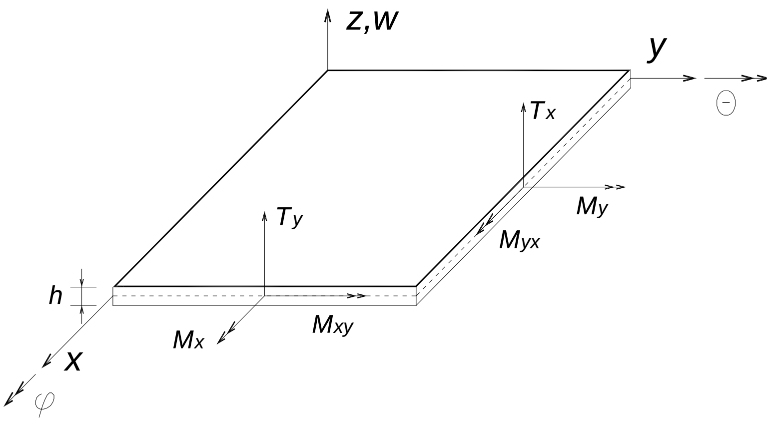

Figure 2 Positive orientations of the forces, deflections and deformations on the plate FE

If we put in the equitation of virtual work (11) the corresponding equations for displacement, deformation and material, it becomes

$$\int_{\Omega} \mathbf{u}^T \mathbf{B}^T \mathbf{D}\,\mathbf{B}\,\mathbf{u}\; d\Omega - \int_{\Omega} \mathbf{u}^T \mathbf{N}^T\; \mathbf{f}\; d\Omega - \mathbf{u}^T \mathbf{s} = 0 \tag{19}$$

Using the matrix operations the above equation in short becomes

$$\mathbf{s}^e = \mathbf{k}^e \mathbf{u} - \mathbf{f}^e \tag{20}$$

where the $\mathbf{k}^e$ and $\mathbf{f}^e$ are defined as

$$\mathbf{k}^e = \int_{\Omega} \mathbf{B}^T \mathbf{D}\,\mathbf{B}\,\mathbf{u}\; d\Omega, \quad \mathbf{f}^e = \int_{\Omega} \mathbf{N}^T\; \mathbf{f}\; d\Omega \tag{21}$$

If the **T** is transformation matrix, than the transformation of the forces and the matrix of the elements from the local to the global coordinate system are described as

$$\begin{aligned} \mathbf{s}_G^e &= \mathbf{s}^e \\ \mathbf{k}_G^e &= \mathbf{T}^T \mathbf{k}^e \mathbf{T} \\ \mathbf{F}_G^e &= \mathbf{F}^e \end{aligned} \tag{22}$$

EXAMPLE

The model is first checked on the patch testes. This example verifies the developed numerical model with the results of the author [2] and the analytic solution [3].

The example deals with the reinforced concrete square slab dimension 10.0m x10.0m, with the thickness of t=0.1m, loaded with the normal concentric force F=100.0kN in the middle of the plate and the Young modulus E=1.2 MPa. The table 1 shows the displacements, while Figures 3 – 5 show the distribution of the bending moments together with their values.

Table 1 Displacement *w* of the middle of the plate

Case	Free standing plate			Fixed plate		
	w	M(υ=0.0)	M(υ= 0.3)	w	M(υ=0.0)	M (υ= 0.3)
Author [2]	12.2	23.06	30.17	6.15	19.32	25.11
Analytic	11.6	-	-	5.59	-	-
LJUSKA (developed software)	12.202(υ=0.0) 11.257(υ=0.3)	16.02	21.038	6.06(υ=0.0) 5.62(υ=0.3)	11.997	15.803
TOWER (comercial software)	10.971(υ=0.0) 10.395(υ=0.3)	19.05	21.600	4.89(υ=0.0) 4.81(υ=0.3)	14.900	16.060

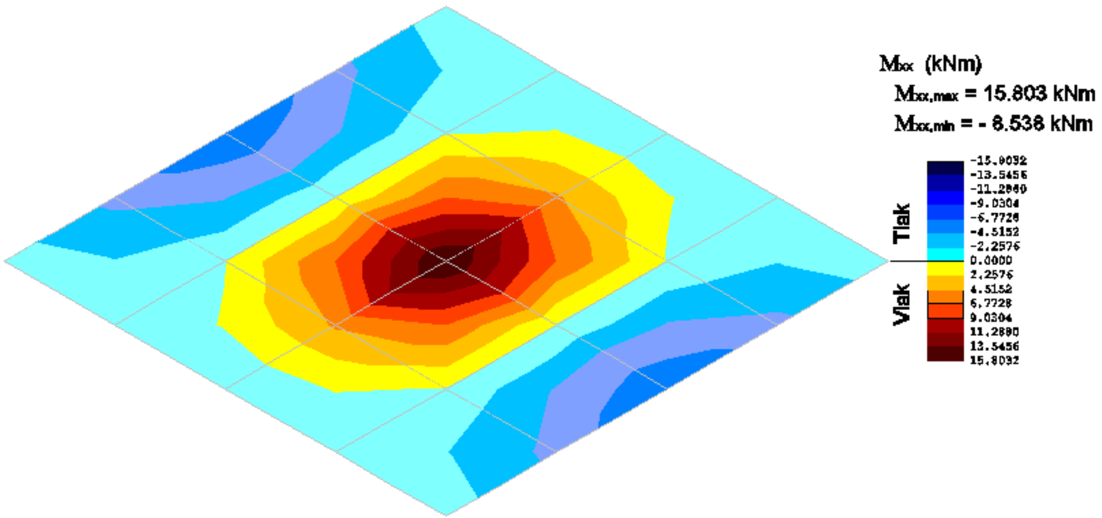

Figure 3 Distribution of the moments M_{xx} in the fixed plate

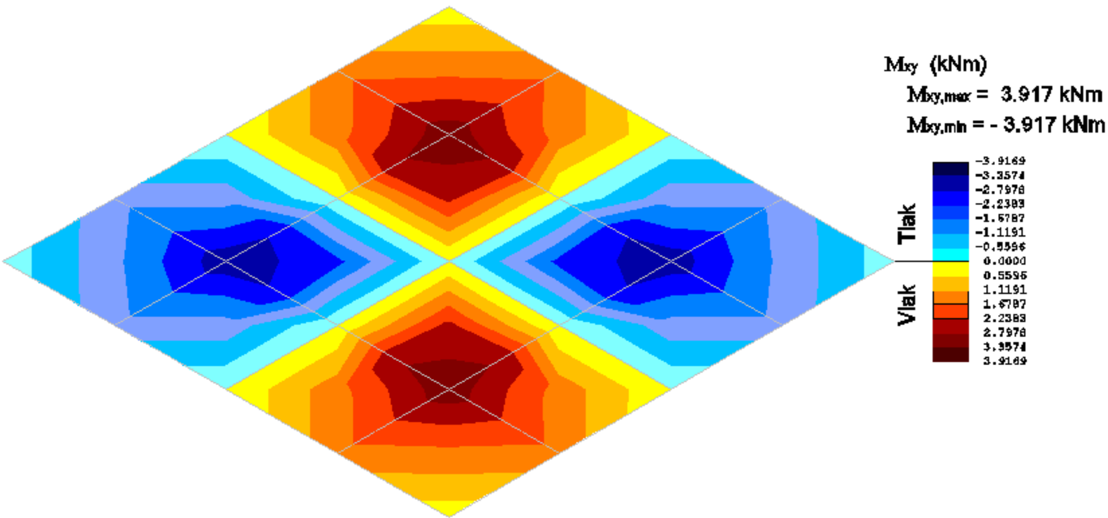

Figure 4 Distribution of the moments M_{xy} in the fixed plate

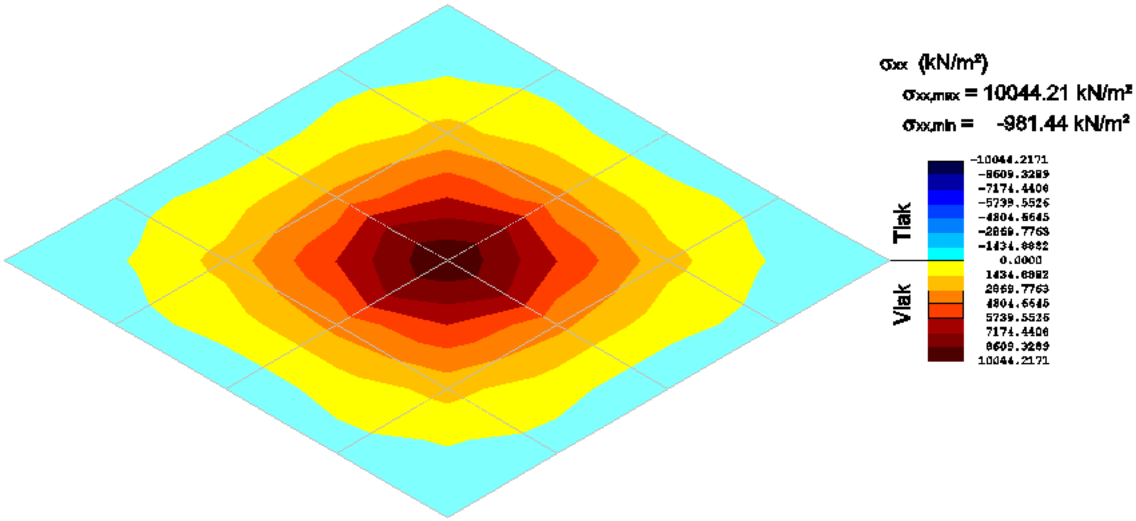

Figure 5 Stress $\boldsymbol{\sigma}_{xx}$ distributions in the fixed plate

CONCLUSION

The developed nonlinear numerical model is applied on the static loaded plate examples shown above. The aim of the example is to show, for relatively rare FE mesh, the good agreement of the deflections between this model, the analytical solution, the other author and the commercial software. Last three figures, which shows the bending moments and stresses, represent the graphical possibility of the developed software.

The maximum deviation of the displacement *w* from the analytical solution (table 1) is under 5%. The deviations of the moment values from the one given by the author [2] are larger, but within the tolerance limit if they are compared with the commercial software Tower. This model doesn't use the total fixing functions, like author [2]. The deviations in generalized forces can be explained with the fact that this model is developed using the displacement method instead of the force method.

REFERENCES

[1] Akmadzic, V. (2008.) Model velikih pomaka u analizi plosnih i linijskih konstrukcija, *Disertation*, Faculty of Civil Engineering University of Mostar, Mostar, B&H

[2] Nikolic, Z. (1999.) Doprinos poboljšanju rješenja nad konacnim elementima linijskih konstrukcija i tankih ploca, *Disertation*, Faculty of Civil Engineering University of Split, Split, Croatia.

[3] Timoshenko, S. P. Gere J. M. (1986.) Theory of elastic stability, McGraw-Hill, London, 1986.

[4] Mihanovic, A. (1993.): Stabilnost konstrukcija, DHGK, Zagreb

[5] Akmadzic, V. Trogrlic, B. and Mihanovic, A. (2010.) Numerical Analysis of beam and surface structures under large displacement, *Proceedings of the 20th DAAAM International Symposium*, Vienna, Austria