"1st International Symposium on Computing in Informatics and Mathematics (ISCIM 2011)"
in Collabaration between EPOKA University and "Aleksandër Moisiu" University of Durrës
on June 2-4 2011, Tirana-Durres, ALBANIA.

Topological Direct Sums of non Convex Barrelled Spaces

Blerina BOÇI[1], Valentina SHEHU[2]

[1] *Department of Mathematic, "Aleksander Moisiu"University of Durres, Albania*
Email:blerinaboci@yahoo.com– Phone:+355693570291
[2] *Department of Mathematic, Tirana University, Albania*
Email:shehuv@yahoo.com– Phone:+355692717219

ABSTRACT

Using the concept of the strings in the vector spaces, is developed a theory related to the topological vector spaces (t.v.s).

At this point of view, there are some important definitions for the strings and we can also see their characteristics in the topological vector spaces.

Also, considering a set of t.v.s we show that the topological product and the topological direct sum coincide if and only if I is finite. We want to show some permanence properties of barrelled spaces and conclude, every t.v.s of second category (i.e a Baire space) is barrelled. Especially (F)-spaces are barrelled. Some important results are: The topological direct sum (the product) of barrelled spaces is barrelled. Every quotient space of a barrelled space is barrelled. Finally, we will show that is also true for a subspace F of finite codimension in the barrelled space E the topology induced on F by E is barrelled.

INTRODUCTION

This study is divided in two sections. The first section deliver the general setting of the theory, topological vector spaces definitions and propositions.

In the second section we investigate the class of "barreled" topological vector spaces. The main part of these sections is taken by theorems on linear mappings

1. Strings and linear topologies, topological direct sums, inductive limits.

Let us begin with some definitions of our principal objects of stady.

In this and the following section we shall only consider vector spaces over the field K of real or complex numbers.

Let E be a vector space over K. A sequence $U = (U_n)_{n\in N}$ of subset U_n of E is colled a string (in E) if

(i) every $U_n \subset U$ is *balanced*, that means for any $x \in U_n$ and $\lambda \in K, |\lambda| \leq 1$, we have $\lambda x \in U_n$,
(ii) every U_n is *absorbing*, that means for any $x \in E$ there is a $\lambda \in K, \lambda > 0$, such that $x \in \lambda U_n$,
(iii) $U = (U_n)_{n\in N}$ is *summative*, that means $U_{n+1} + U_{n+1} \subset U_n$ for all $n \in N$.

U_1 is called the beginning of the string U and U_n is the nth knot of U.

If the vector space E is equipped with a topology τ, we denote this topological space by (E,τ). τ is a linear topology, if addition and scalar multiplication are continuous mappings from $E \times E$ and $E \times K$ into E. If τ is linear and Hausdorff, we call (E,τ) a topological vector space (abbreviated: t.v.s) .

Definition 1.1

A topological space (E,τ) is said to be Hausdorff topological space if and only for any pair of distinct point $x, y \in E, (x \neq y)$, there exist sets $U, V \in \tau$ such that $x \in U, y \in V$ and $U \cap V = \emptyset$

Definition 1.2

Let Φ be a set of strings in a vector space E, such that for all $U, V \in \Phi$ there is a $W \in \Phi$ with $W \subset U \cap V$.

A set Φ of strings in E with this property is called directed.

Let τ be a linear topology on E. A string $U = (U_n)_{n\in N}$ in (E,τ) is called a *topological string* , if every knot U_n is a neighbourhood of 0.

Theorem 1.3

Let Φ be a directed set of stings, there exist a linear topology τ_Φ such that, the knots of the strings in Φ form a base of 0-neighbourhoods in (E,τ).

A directed set Φ of strings in a t.v.s (E,τ) with $\tau = \tau_\Phi$ is called *fundamental* i.e the knots of the strings in Φ form a base of 0-neighbourhoods in (E,τ). In this case we say: Φ *generates* τ_Φ.

Example. The set of all strings in a vector space E generates a linear topology $\tau^{\mathcal{F}}$ on E, that is the finest linear topology on E. Every absorbing absolutely convex set U in E gives a string $\forall\, n \in N, U_n = \frac{1}{2^{n-1}} U$. $\mathcal{U} = \{U_n : n \in N\}$ *is called natural string of* U.

The set of all natural trings in E generates a locally convex topology τ^{C} on E and τ^{C} is the finest locally convex topology on E. It's clear that $\tau^{C} \subset \tau^{\mathcal{F}}$. If we collect the last results, we have:

Theorem 1.4

If dim E is countable, then $\tau^{\mathcal{F}} = \tau^{C}$. If dim E is uncountable, then $\tau^{\mathcal{F}}$ is strictly finer than τ^{C}

Theorem 1.5

Let E be a vector space. For $i \in I$, I a index set, let $A_i : E_i \to E$ be a linear mapping from the t.v.s (E_i, τ_i) into E. Assume $E = \sum_{i \in I} A_i(E_i)$ and consider the set of srings in E given by

$$\Phi = \{U : U \text{ is a string } E \text{ and } A_i^{-1}(U) \text{ is a topological string in } (E_i, \tau_i)\ \forall\, i \in I\}$$

Then we have

(i) is directed.

(ii) The topology τ_Φ is the finest linear topology on E such that all mappings $A_i, i \in I$, are continuous.

The topology τ_Φ is called the inductive topology on E with respect to (E_i, τ_i, A_i). A t.v.s constructed as in the theor. 1.5 is called the inductive limit of the spaces (E_i, τ_i) with respect to A_i and we denote it by $(E, \tau) = \sum_{i \in I}(E_i, \tau_i, A_i)$.

Example. For each $i \in I$ let (E_i, τ_i) be a t.v.s. and denote by E the algebraic direct sum of the E_i, $E = \bigoplus_{i \in I} E_i$. Let τ be the inductive limit topology on E with respect to the embeddings $I_i : (E_i, \tau_i) \to E$. Since $E \subset \prod_{i \in I} E_i$, and since all $I_i : (E_i, \tau_i) \to (E, \tau_\Pi)$ are continuous (τ_Π denotes the topology which is induced by $\prod_{i \in I}(E_i, \tau_i)$ on E), we have $\tau_\Pi \subset \tau$, and the inductive limit topology on E is Hausdorff.

(E, τ) is colled topological direct sum of the (E_i, τ_i), and we denote this sum $(E, \tau) = \bigoplus_{i \in I}(E_i, \tau_i)$.

Let $(E, \tau) = \bigoplus_{i \in I}(E_i, \tau_i)$ a topclogical direct sum and τ_Π denotes the topology which is induced by $\prod_{i \in I}(E_i, \tau_i)$ on E, then we have $\tau_\Pi < \tau$.

Theorem 1.6

For a finite subset I' of I the topologies τ and τ_Π coincide on $\bigoplus_{i\in I} E_i$ if $(E,\tau) = \bigoplus_{i\in I}(E_i,\tau_i)$.

2. Barrelled topological vector spaces

In every t.v.s (E,τ) there is a fundamental set of closed strings (that are strings, whose knots are closed). But in general not every closed string in (E,τ) is a topological string.

Example. We consider the space of all finite sequences as a subspace of the space of all sequences, endowed with the usual product topology. The set $U = \{(x_i)_{i\in N}: x_i \in \omega, \left||x_i\right| \le 1,\ i \in N\}$ is absolutely convex, absorbing and closed in . The natural string $U = (U_n)_{n\in N}$ of U with $U_n = \frac{1}{2^{n-1}}U$ is a closed string in , but the U_n are not 0-neighbourhoods on the topology induced on by .

Definition 2.1

A t.v.s (E,τ) is called barrelled if all closed strings in (E,τ) are topological string.

Every t.v.s of second category (i.e a Baire space) is barrelled. Especially F-spaces are barrelled. (A metrizable complete t.v.s. is called F-space).

Examples. (i) Let $U = (U_n)_{n\in N}$ be a closed string in (E,τ). For fixed $n \in N$ we have $E = \bigcup_{k=1}^{\infty} kU_{n+1}$. (E,τ) is of second category and so U_{n+1} has an inner point. Since $U_{n+1} + U_{n+1} \subset U_n$, the knot U_n has 0 as inner point and is a neighbourhood of 0. Hence U is topological.

(ii) Let Σ be a σ-algebra of subsets of a set X and μ a measure on Σ. Let $\varphi(t)$ for $t \ge 0$ be a continuous, increasing, nonnegative function with: $\varphi(t) = 0$ if and only if $t = 0$, $\varphi(2t) \le k\varphi(t)$ for a certain k and all t. We denote by

$$L^\varphi(X,\Sigma,\mu)$$

the linear space of all μ-equivalence classes f of measurable functions on X with

$$\int_X \varphi(|f(t)|)d\mu < \infty.$$

For certain ε_n the sets $U_n = \{f: \int_X \varphi(|f(t)|)d\mu < \varepsilon_n\}$ give us a string in $L^\varphi(X,\Sigma,\mu)$ and generate a metrizable linear topology on these spaces. Moreover the $L^\varphi(X,\Sigma,\mu)$ are F-spaces with these topologies.

Choosing X, Σ, μ in a concreter form one obtains many well known F-spaces:

If $\varphi(t) = t^p, 0 < p < \infty, X = [0,1]$ the unit intervall, Σ the Lebesgue measurable subsets and μ the Lebesgue measure, then

$L^\varphi(X, \Sigma, \mu) = L^p(0,1)$.

This space is Banach space for $1 \leq p < \infty$, hence locally convex. For $0 < p < 1$ it is not locally convex.
The closed strings in a t.v.s (E, τ) generate a linear topology on E. We call this topology the *strong topology* of (E, τ) and denote it by τ^b. We have $\tau \subset \tau^b$.

Theorem 2.2
Let (E, τ) be a t.v.s $\Phi = \{U = (U_n)_{n \in N} : U \text{ a closed string on } (E, \tau)\}$ then we have
(i) is a directed set of strings.
(ii) $\tau < \tau_\Phi$

We want to show some permanence properties of barrelled spaces.

Proposition 2.3
If $(E, \tau) = \sum_{i \in I}(E_i, \tau_i, A_i)$ is the inductive limit of the barrelled spaces (E_i, τ_i), then (E, τ) is also barrelled.
A consequence of theor. 2.3 is

Proposition 2.4
The topological direct sum of barrelled spaces is barrelled. Every quotient space of a barrelled space is barrelled.
A result similar to prop. 2.3 for projective limits does not hold as the example at the beginning of this section shows. However it is shown that the product of barrelled spaces is barrelled.
A bounded absolutely convex subset K of (E, τ) is called Banach disk, if E_K is a Banach space with the norm generated by K. Special Banach disks are compact absolutely convex sets.

Proposition 2.5

If $(E, \tau) = \prod_{i \in I}(E_i, \tau_i)$ then $(E, \tau^b) = \prod_{i \in I}(E_i, \tau_i^b)$

As a corollary we obtain

Corollary 2.6

The product $\prod_{i \in I}(E_i, \tau_i)$ of barrelled spaces (E_i, τ_i) is barrelled.

Let F now be a linear subspace of the t.v.s (E, τ) and let $\hat{\tau} = \{G \cap F : G \subset \tau\}$ denote the topology induced on F by τ.

Proposition 2.7

Let F be a linear subspace of finite co dimension (i.e $E = F \oplus G$, $codF = dimG$) in the barrelled space (E, τ), then $(F, \hat{\tau})$ is barrelled.

Is also true for subspaces of countable co dimension.

Conclusion

Starting from the notion of a "string" in a vector space we develop a general theory of topological vector spaces giving most of the results known up to now. The importance of the strings can be seen from the very beginning. They help to develop a theory of topological vector spaces which gives a satisfactory generalization of the locally convex theory. In this general situation there is no sufficiently large topological dual space. To see this one has to find new ways and proofs, proofs which are free from duality theory. Of course, these new proofs can also be used to prove the known results and even new ones in the locally convex.

REFERENCES

[1] F.Albiac and N:J.Kalton Springer 2006 "Topics in Banach Spaces Theory] J. Tinsley Oden 1999 "Applied Functional Analysis"
[2] N. Adasch 1978 "Topological Vector Spaces"
[3] R. E. Edwards 1995 "Functional Analysis Theory and Aplications"
[4] Francois Treves 2006 " Topological Vector Spaces,Distributions and Kernels"
[5] Article: 'Sequentially Barrelled Spaces and the Strongest Locally Convex Topology' Jerzy Kakol