Minimal Quasi-Ideals In d-Semigroup

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ABSTRACT

In this paper we will prove some theorems that discover the structure of minimal quasi-ideals in Γ-semigroups without zero. The main result is these are Γ-subgroups. We will prove also that this structure is not true for respective quasi-ideal of Γ-semigroup without zero, living opened the problem of conditions that must satisfy a Γ-semigroup without zero in order to have, in this case, an analogous structure.

***Keywords**: d-semigroup, right (left) principal ideal, quasi-ideal, idempotent element, minimal quasi-ideal, principal quasi-ideal.*

INTRODUCTION

Definition 1.1 **[1] Let be *M* = {*a*, *b*, *c*, ..., } and Γ = {*x*, *y*, *z*, ..., } two nonempty set. We call the set *M* a Γ-semigroup if:**
1. *axb* Č *M*,
2. (*axb*) *yc* = *ax* (*byc*) for *a*, *b*, *c* Č *M* and *x*, *y* Č Γ.

In this definition so called multiplication, which is similar with multiply of an ordinary semigroup is described intuitively, so let's try to precise it.

Let *M* = {*a*, *b*, *c*, ..., } and Γ = {*x*, *y*, *z*, ..., } be two nonempty sets. The multiplication in the set *M* by elements of Γ set, which are between, we call every map of $M \times \Gamma \times M$ in *M*. This multiplication we call also Γ-multiplication in *M* and will denote by $(\cdot)_\Gamma$. Result of Γ- multiplication in *M* for every two elements *a*, *b* of *M* and every element $x \in \Gamma$, which is the image of mapping $(\cdot)_\Gamma$ for triple (*a*, *x*, *b*), $(\cdot)_\Gamma(a, x, b)$, will denote simply by *axb*.

Definition 1.1'. **[3] We call *I* proper ideal if it is a right (resp. left, two-sided) ideal of d-semigroup *M* and a proper subset of *M*.**

Now, in a Γ-semigroup *M* we may define analog relations of Green’s relations in a semigroup [4].

Let *M* be a Γ-semigroup and *a*, *b* from *M*. We define binary relations in the following way:

(1) $a\, \mathcal{L}\, b \Leftrightarrow (a)_l = (b)_l$.

(2) $a\, \mathcal{R}\, b \Leftrightarrow (a)_r = (b)_r$.

(3) $a\, \mathcal{J}\, b \Leftrightarrow (a) = (b)$.

(4) $a\, \mathcal{H}\, b \Leftrightarrow a\, \mathcal{L}\, b \wedge a\, \mathcal{R}\, b$

(5) $a\, \mathcal{D}\, b \Leftrightarrow \exists\, c \in M, a\, \mathcal{L}\, c \wedge c\, \mathcal{R}\, b$.

Note that relation $\mathcal{L}$, $\mathcal{R}$, $\mathcal{H}$, $\mathcal{J}$ are equivalence relations in *M*. The classes of equivalence of an element *a* from Γ-semigroup *M* by these equivalence relations we denote respectively by:

L_a, R_a, H_a, J_a.

Theorem 1.2. (**Green’s Theorem**) **[6]. If *H* is a $\mathcal{H}$–class of Γ-semigroup *M*, then either for all *x* from Γ, $HxH \cap H = \Phi$ or *H* is a subgroup of M_x.**

Definition 1.3. [4] An element *e* Č *M* is called idempotent in a Γ-semigroup *M* if there is *x* Č Γ such that *exe* = *e*.

Definition 1.4. The quasi-ideal of Γ-semigroup *M* we call a nonempty subset *Q* of *M* such that

$$Q\,\Gamma M \cap M\,\Gamma Q \subseteq Q.$$

It is clear that every left (resp. right, two-sided) ideal of a Γ- semigroup *M* is quasi-ideal of *M*. It is clear also that every quasi-ideal *Q* of *M* is Γ-subsemigroup. A Γ-semigroup may have not a proper quasi-ideal. As result of the absence of the proper quasi-ideals of a Γ-semigroup we have an important property. So, is true the following theorem:

Theorem 1.5. Let *M* be a Γ-semigroup and *x* Č *M* be a fixed whatever element. M_x is group if and only if *M* have not proper quasi-ideals.

Theorem 1.6. If *e* = *exe* (*e* Č *M*, *x* Č Γ) is idempotent element of Γ-semigroup of *M* and *L*, *R* are left, right ideals of *M*, respectively, then:

$$Rxe = R \cap Mxe,$$
$$exL = L \cap exM,$$

are quasi-ideals of *M*.

Definition 1.7. Let A be a nonempty subset of a Γ-semigroup M. The quasi-ideal generated by A will call intersection of all quasi-ideals $(A)_q$ of M that contain A.

Definition 1.8. If $A = \{a\}$, then quasi-ideal $(\{a\}_q)$, which denoted by $(a)_q$, we will call principal quasi-ideal generated by element a of M.

The following proposition give the structure of quasi-ideal generated by a nonempty subset $A \neq \Phi$ of a Γ-semigroup M.

Theorem 1.9. If A is a nonempty subset of a Γ-semigroup M, then (A Ć M ΓA) ∩ (A Ć M ΓA) is quasi-ideal $(A)_q$ of M, generated by A.

2. Main Results

The minimal quasi-ideals in Γ-semigroups without zero are discussed before, but without discover their Γ-groupoid structure, which will be obtained through Greens relations.

Definition 2.1. A quasi-ideal Q of a d-semigroup without zero M is called minimal if Q not hold in proper quasi-ideals of this d-semigroup.

The following theorem characterizes the minimal quasi-ideal:

Theorem 2.2. A quasi-ideal Q of a d-semigroup M is minimal if and only if it is a interception of the minimal left ideal L and minimal right ideal R, i.e. $Q = L \cap R$.

Theorem 2.3. A quasi-ideal Q of a d-semigroup M is minimal if and only if Q is a $\mathcal{H}$ -class.

Proof. Let suppose Q is a minimal quasi-ideal and $a \in Q$. Since

$$(a)_q = a \cup (M\,\Gamma a \cap a\,\Gamma M)$$

we have:

$$(a)_q \subseteq Q \cup (M\,\Gamma Q \cap Q\,\Gamma M) \subseteq Q$$

and since Q is minimal, $(a)_q = Q$.
Conversely, let have

$\forall\, a \in Q, (a)_q = Q.$

If Q' is a quasi-ideal of M, such that $Q' \subseteq Q$, for an $a \in Q'$ we have $Q = (a)_q \subseteq Q'$, hence $Q = Q'$. Thus, Q is minimal. So, we have prove the equivalence:
Q is minimal if and only if for every a Č=Q, $(a)_q$ =Z Q.
From this we get Q is a $\mathcal{H}$ -class. ■

Theorem 2.4. A quasi-ideal Q of a d-semigroup without zero M is minimal if and only if Q is d-subgroup of M. Moreover, every minimal quasi-ideal Q of M, for all x Č d holds equalities:

$Q = exM$ ą $Mxe = exMxe$,

where $e = exe$ is the unity element of subgroup Q^x.

Proof. Let suppose Q is a minimal quasi-ideal of Γ-semigroup M, then from **Theorem 2.3** Q is a $\mathcal{H}$ –class. If $\mathcal{H}$-class Q we denote by H, then for $x \in \Gamma$, $Q^x = H$ is a Γ-subgroup because have not proper quasi-ideals of M.
Conversely, let suppose for $x \in \Gamma$ the quasi-ideal Q is a Γ-subgroup. If Q' is a quasi-ideal of Γ-semigroup M such $Q' \subseteq Q$, then:

$$Q'\Gamma Q \cap Q\Gamma Q' \subseteq Q'\Gamma M \cap M\Gamma Q' \subseteq Q',$$

which shows Q' is a quasi-ideal of Q.

We know Q is a Γ-subgroup, then it not contain proper quasi-ideals, thus $Q' = Q$, which imply Q is minimal quasi-ideal.

Let Q be a minimal quasi-ideal of Γ-semigroup M and $e = exe$, where $e \in Q$ is unity of Γ-subsemigroup Q^x, for every $x \in \Gamma$. From **Theorem 2.2** Q is the intersection of a right minimal ideal R and a left minimal ideal L and since $e \in Q = R \cap L$ imply $e \in R$ and $e \in L$. Since:

$$exM \subseteq R\,\Gamma\, \text{j} \subseteq R,$$

then from minimality of R we have

$$R = exM.$$

Thus, analogously, we show $L = Mxe$. Now we have:

$$Q = R \cap L = exM \cap Mxe.$$

Finally, we prove

$$exM \cap Mxe = exMxe.$$

Let $a \in Q = exM \cap Mxe$. Then we have $a = exm = nxe$ where $n, m \in$ j and thus:

$$a = (exe)xm = ex(exm) = ex(nxe) \in exMxe.$$

So

$$exM \cap Mxe \subseteq exMxe.$$

Conversely, let suppose $a \in exMxe$, then:

$$a = exbxe = ex(bxe) \in exM$$

and

$$a = exbxe = (exb)xe \in Mxe,$$

then $a \in exM \cap Mxe$, thus we find

$$exMxe \subseteq exM \cap Mxe.$$

From this inclusion and above conversely inclusion we have equality. ■

Naturally, rise question: Are hold true the above theorem if we substitute the Γ-semigroup without zero by a Γ-semigroup with zero, certainly demanding that quasi-ideal Q of Γ-semigroup with zero M not contain quasi-ideals other than zero and M quasi-ideal as well as by substitute the requirement Γ-group have zero?

The response of this question is negative, as demonstrate the following counterexample:

Counterexample 1

Let M be the set of complex numbers C and Γ = C. Let define the Γ-multiplication in C by common multiplication axb for every two complex numbers a, b and for every $x \in \Gamma$. It is clear $(M, (\cdot)_\Gamma)$ is Γ-semigroup with zero. The set M is a quasi-ideal. The quasi-ideal M does not contain anyone quasi-ideal other than zero and *M*. Indeed, if *Q* is a quasi-ideal different from zero, then for a element *a* not equal zero and for every element $b \in M$ we have element $ba^{-1} \in M$ and $1 \in \Gamma$ for which holds true equalities:

$$b = ba^{-1} \cdot 1 \cdot a = a \cdot 1 \cdot a^{-1} b .$$

These equalities show $b \in Q$, since *Q* is quasi-ideal of *M*. So, *Q* = *M*.
The minimal quasi-ideal *M* is not Γ-group with zero because for $x = 0 \in \Gamma$, we have *MxM* = 0 and thus *M* is not a group with zero of semigroup with zero M_0.

But, there are Γ-semigroups with zero for which hold true the analogous theorem for Γ zero (C, $(\cdot)_{\mathbb{C}^*}$) is Γ-group because for every complex number *x* diverse from zero, the set C*x*C përputhet me *x* and so it is group with zero of the semigroup with zero $\mathbb{C}_0$.

Considering situation for Γ-semigroup with zero we let opened this problem:

What conditions must satisfy a d-semigroup with zero to hold true the analogous theorem of Theorem 2.4 for d-semigroup with zero?

The minimal quasi-ideals in Γ-semigroupd without zero are discussed before by other authors, but fail to show their groupoid structure, which will be obtained through Greens relations.

Theorem 2.5. If *e = exe, x* Č⁼d, is a idempotent element of a d-semigroup *M* contained in a minimal left ideal *L*, then *exL* is d-group, and moreover is a minimal quasi-ideal of *M*.

Proof. Let *e* be an idempotent, *e* = *exe*, $x \in \Gamma$, contained in left ideal *L*. Let consider *exL*. We will show *G* = *exL* is a subgroup of M_x. Take elements *exm*, $exn \in exL$, where $m, n \in L$. For every $y \in \Gamma$ are true the following equalities:

$$(exm)\, y\, (exn) = ex[my(exn)\,] = ex[(mye)\, xn]$$

showing *G* is Γ-subsemigroup of Γ-semigroup *M*. It is clear that *e* = *exe* is left unity element of semigrup (*G*, ○), where ○ is the induced operation from operation of semigroup M_x. If *g* = *exl* is an arbitrary element of semigroup (*G*, ○), then *Lxg* is the left ideal of *M*, because:

$$M\,(\Gamma Lxg) \subseteq (M\,\Gamma L)\, xg \subseteq Lxg$$

Since:

$$Lxg = Lx\,(exl) = (Lxe)\,xl \subseteq M\,\Gamma L \subseteq L$$

and L is a minimal left ideal of M we have $Lxg = L$. From this we find

$$(ex)Lxg = exL,$$

or, same

$$(exL)\,xg = exL.$$

So, the element $g = exl$ of semigroup $(G, \circ)$ has as left inverse element an element exl_1 of exL. Thus, semigroup $(G, \circ)$ is group. Γ-semigroup $(G, (\cdot)_\Gamma)$ is Γ-group, due to a theorem, and therefore is Γ-subgroup of Γ-semigroup $(M, (\cdot)_\Gamma)$ if with Γ-subgroup of the Γ-semigroup $(M, (\cdot)_\Gamma)$ we mean every Γ-semigroup such that Γ-semigroup generated by it is Γ-group. Now, basing on **Theorem 2.4** we get out exL is a minimal quasi-ideal of M. ■

Theorem 2.6. If *e* = *exe*, *x* Č d is an idempotent contained in a right minimal ideal *R* of a d-semigroup *M*, then *Rxe* is a dJgroup and, moreover, a minimal quasi-ideal of *M*.

Proof. Let $e = exe$ be an idempotent element of Γ-semigroup M, wich is an element of right ideal R. Let show that $G = Rex$ is a subgroup of M_x. Let take elements mxe, nxe of Rxe. For every $y \in \Gamma$ are true the following equalities:

$$(mxe)\,y\,(nxe) = mx\,[(eyn)xe\,] = [mx(eyn)]ex,$$

which show that G is a Γ-subsemigroup of Γ-semigroup M. It is clear that $e = exe$ is the left unity element of semigroup $(G, \circ)$, where $\circ$ is the induced operation from operation of semigroup M_x. If $g = rxe$ is an arbitrary element of G, then gxR is a right ideal of M, because:

$$(gxR)\,\Gamma M \subseteq gx(R\,\Gamma M) \quad \subseteq \; gxR$$

whiles:

$$gxR = (rxe)xR = rx(exR) \subseteq R\,\Gamma M \subseteq M.$$

So, while R is minimal right ideal of M we have $gxR = R$. From this we get

$$(gxR)xe = xeR,$$

or, similarly

$$gx(Rex) = Rxe.$$

Thus, the element $g = rxe$ of semigroup $(G, \circ)$ has as left inverse element an element r_1xe of Rxe. So, semigroup $(G, \circ)$ is group. From this we get Γ-semigroup $(G, (\cdot)_\Gamma)$ is Γ-group and hence G is Γ-subgroup. From this, basing on **Theorem 2.4** we have Rxe is also a minimal quasi-ideal of M.

Theorem 2.7. If d-semigroup without zero *M* has a reducible element, contained in a minimal quasi-ideal *Q* of *M*, then *M* is a d-group.

Proof. Basing on **Theorem 2.4** Q is a minimal quasi-ideal of Γ-semigroup j and thus it is a Γ-subgroup. Let $e = exe$ be the inverse element of Q^x and $a \in$ j a reducible element in Q for the element a we have $exa = a$. If $m\in$ j is an arbitrary element, we find that:

$$mx\,(exa) = mxa \text{ ose } (mxe)\,xa = mxa.$$

Its imply $mxe = m$. In the similar manner we show that $exm = m$. So, e is the identity element for semigroup $(M, \circ)$. Since $e \in Q$, if $m \in$ j : we have:

$$M = mxe = exm \in M\Gamma Q \cap Q\Gamma M \subseteq Q.$$

It imply $M = Q$. Thus, M is Γ-group. ■

Remark

It seems that removing of idempotent element in **Theorem 2.3** is replaced by inserting a reducible element but we may prove every idempotent in a minimal quasi-ideal Q is a reciprocal reducible element. For this is suffice to prove:

$$exa = exb \Rightarrow a = b,$$
$$axe = bxe \Rightarrow a = b.$$

Since e is an idempotent in minimal quasi-ideal Q, then $(Q^x, \cdot)$ is group, where e is its unity and then we will have:

$$a = exa = (exe)\,xa = ex\,(exa) = ex\,(exb) = (exe)\,xb = exb = b.$$

This prove *e* is the left reducible. Similarly, we prove also that *e* is right reducible.
We emphasize that existence of idempotent element in a minimal quasi-ideal of a
Γ-semigroup is a necessary condition for it to be a group.

Theorem 2.8. Let *Q* be a quasi-ideal of d-semigroup without zero *M*. *Q* is minimal if and only if for *q* Č-*Q* we have:

$$\boldsymbol{Q \,\mathrm{d}\, q \cap q \,\mathrm{d}\, Q = Q = (a)_q} \,.$$

Proof. Let suppose *Q* is a quasi-ideal of *M*, which satisfy the equality. Let Q' be a quasi-ideal of Γ-semigroup j and $Q' \subseteq Q$.
On the other hand, let *q* be an element of Q'. Then:

$$Q = q\,\Gamma Q \cap Q\,\Gamma q \subseteq Q'\Gamma M \cap M\,\Gamma Q' \subseteq Q'.$$

So, $Q = Q'$ and consequently *Q* is a minimal quasi-ideal.
Conversely, let suppose *Q* is a minimal quasi-ideal of *M* and $q \in Q$ is an arbitrary element of *Q*, then, firstly we will prove $Q\,\Gamma q \cap q\,\Gamma Q$ is a quasi-ideal of *M*.
We know $Q = (a)_q$. Since $q\,\Gamma q \in Q\,\Gamma q \cap q\,\Gamma Q$ then it is nonempty set. Moreover:

$$Q\,\Gamma(Q\,\Gamma q \cap q\,\Gamma Q) \cap (Q\,\Gamma q \cap q\,\Gamma Q)\,\Gamma Q \subseteq Q\,\Gamma(Q\,\Gamma q) \cap (q\,\Gamma Q)\,\Gamma Q \subseteq$$
$$\subseteq (Q\,\Gamma Q)\,\Gamma q \cap q\,\Gamma Q\,\Gamma Q) \subseteq Q\,\Gamma q \cap q\,\Gamma Q.$$

So, $Q\,\Gamma q \cap q\,\Gamma Q$ is a quasi-ideal of *Q* and since it is minimal then

$$Q\,\Gamma q \cap q\Gamma Q = Q = (a)_q.$$

CONCLUSION

In this paper we are proved some theorems that help to find out the structure of minimal quasi-ideals in Γ-semigroups without zero. The our aim was to discover their structures, which, as we proved, is the same with Γ-subgroups. But this structure was not true for respective quasi-ideal of Γ-semigroup without zero, living opened the problem of conditions that must satisfy a Γ-semigroup without zero in order to have, in this case, an analogous structure.

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