Prime Ideals And Bi - Ideals In Gamma Near– Rings

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ABSTRACT

Throughout this paper we introduce the concept of prime ideals, maximal ideals and bi – ideals in Γ - near - rings obtaining some characterizations and their links. We introduce that if M is Γ - near –ring which for a ¥ ∈ M exists an element which is ¥ - unit then every maximal ideal I of M is prime ideal.

INTRODUCTION

Let's consider M and Γ as two non-empty sets. Every map of M x Γ x M in M is called Γ-multiplication in M and is denoted as $(.)_\Gamma$. The result of this multiplication for elements a, b ∈ M and ¥ ∈ Γ is denoted a¥ b.

According to Satanarayana [2], Γ- near-ring is a classified ordered triple

$(M, +, (\cdot)_\Gamma)$ where M and Γ are non empty sets, + is a addition in M, while $(\cdot)_\Gamma$ is

Γ - multiplication on M satisfying the following conditions:

(M, +) is a group.

$\forall$ (a, b, c, α, β) ∈ $M^3 x\Gamma^2$, (aαb)βc = aα(bβc).

$\forall$ (a, b, c, α,) ∈ $M^3 x\Gamma$, (a + b)αc = aαc + bαc.

Example 1. [2]. Let (G, +) be a group, X a non empty set and M a set of all the mappings of X in G. The ordered pair (M, +), where + is a addition of mappings of X in G defined by the equality:

$$(f + g)(x) = f(x) + g(x)$$

is a non abelian group when G is non abelian. Let Γ be a set of all the mappings of G and X. If the product of f¥ g is defined by f o¥ o g for every f, g ∈ M and every

¥ ∈ Γ, then it is defined in M a Γ - multiplication, $(\cdot)_\Gamma$ such as for every three elements f_1, f_2, f_3 of M and every two elements α, β of Γ the equalities are true:

$$f_1\alpha(f_2\beta f_3) = (f_1\alpha f_2)\beta\, f_3,$$

$$(f_1 + f_2)\alpha f_3 = f_1\alpha f_3 + f_2\alpha f_3.$$

Consequently, $(M, +, (\cdot)_\Gamma)$ is Γ - near-ring.

Example 2. If in **example 1** the set X is the retainer of G' of group (G',+), M is the set of all the mappings of G in G' such as f(0) = 0 and Γ is the set of all the mappings of G'in G, again M is a Γ-near-ring in relation to the addition of mappings element per element and

Γ-multiplication is defined by the general composition fo¥og for every two elements f, g of M and every element¥ ∈ Γ.

PRELIMINARY CONCEPTS

Definition 2.1. [3] *Ideal P of* d*-near-ring (M, +, (ê̆)*$_d$*) is called prim, or prime if for every two ideals I, J of M it is true the implication :*

I d *J* ċ *P* Ĝ *I* ċ *P* Ę *J* ċ *P.*

If M_1 is a subgroup different from the empty subgroup Φ, then the intersection of all the ideals that hold M is the smallest ideal that holds M_1 and it's called *the ideal that derives from the subgroup* M_1 and it's denoted (M_1). If $M_1 = \{a\}$, then $(\{a\})$ is called *primary ideal derived from the element a* Č *M* and it is denoted simply (*a*).

Definition 2.2. [1] *A* d*-near-ring (M, +, (ê̆)*$_d$*) it's called prime if zero ideal {0} = 0 is prime ideal.*

Definition 2.3. *A* d*-near-ring (M, +, (ê̆)*$_d$*) is called prime if there are no other ideals except the zero ideal, 0 = {0} and M, which are called not proper ideals, meanwhile every other ideal different from them is called proper ideal of M.*

Definition 2.4. *Ideal I of* d*-near-ring M it's called maximal ideal if I ÷ M and for every J of M, I* ċ *J* Ĝ *J = M* Ę *J = I.*

It's very clear that ideal *I* of Γ-near-ring $(M, +, (\cdot)_\Gamma)$ is maximal ideal only when it is the maximal element of the group ideals of M different from M itself.

Definition 2.5. *Prime minimal ideal of ideal I of* d*-near-ring M it's called every minimal element in the prime ideals group that containing ideal I ordered by inclusion.*

Here we will give concepts and we will present the same auxiliary propositions, which we will use further in the presentation of the main results of the proceeding.

Let (M, +, $(\cdot)_\Gamma$) be a Γ-near-ring and A, B two subsets of M. We define the set

AΓB = {a¥ b ∈ M / a, b ∈ M and ¥ ∈ Γ }.

For simplicity we write aΓB instead of {a}ΓB and similarly AΓb instead of AΓ{b}.

Also for every ¥ ∈ Γ we define

A¥ B = {a¥ b ∈ M / a, b ∈ M}

and for simplicity we write a¥ B and A¥ b respectively instead of {a}¥ B and A¥ {b}.

In [4] is define the set as well as

AΓ∗ B = {a ¥ (a' + b) - a¥ a' / a, a' ∈ A, ¥ ∈ Γ, b ∈ B}

Definition 2.6. [6]. *A* d*-near-ring M is called zero – symmetric if for every a* Č *M and for*

every ¥ Č d *we have a¥ b = 0.*

Γ -near-ring of **example 2** is Γ - near-ring zero – symmetric, whereas the one of **example** in general is not zero-symmetric.

Definition 2.7 [4]. *Let (M, +, (ȅ)*$_d$*) be a* d*-near-ring. A subgroup B of group*

*(M, +) is called bi-ideal of M if B*d*M*d*M* ą *(M*d*M)*d∗ *B* ċ *B.*

Definition 2.8. *A* d*-near-ring is called B-simple if there are no bi-ideal different from*

zero and from M.

Definition 2.9. *A bi –ideal B of* d *- near-ring is called minimal if it is different from*

zero and it doesn't contain any bi-ideal different from zero or from B itself.

Theorem 3.1 *Let P be an ideal of* d-*near-ring M. The following conditons are equivalent:*

1) Ideal P is prime.

2) For every two ideals I, J of M we have the implication:

$I \not\subseteq P$ ė $J \not\subseteq P$ Ĝ I d $J \not\subseteq P$.

3) For every two elements i, j of M, ? *(i, j)* Č M^2, *i* č *P* ė *j* č *P* Ĝ *(i)*d*(j)* ĉ *P.*

4) For every two ideals I, J of M we have P Ċ *I* ė *P* Ċ *J* Ĝ $I/J \not\subseteq P$.

Proof. The equivalence **1)** $\Leftrightarrow$ **2)** is very clear.

2) $\Rightarrow$ **3)** We suppose that **2)** is true. Let i, j be to elements of M such that $i \notin P$ and $j \notin P$. Hence $(i) \not\subset P \wedge (j) \not\subset P$ and therefore from **2)**, $(i)\,\Gamma\,(j) \not\subset P$.

3) $\Rightarrow$ **4)** If $P \subset I$ and $P \subset J$ we find the element $i \in I \backslash P$ and element $j \in J \backslash P$. So, $i \notin P \wedge j \notin P$ and therefore due to **3)** follows that $(i)\Gamma(j) \not\subset P$.

4) $\Rightarrow$ **2)** We suppose that the proposition **4)** is true. If $I \not\subset P \wedge J \not\subset P$, then elements $i \in I \backslash P$ and $j \in J \backslash P$ exist. Hence we have

$$P \subset (i) + P \wedge P \subset (j) + P$$

And therefore due to **4)** ,

$$((i) + P)\Gamma((j) + P) \not\subseteq P.$$

Hence, elements $i' \in (i)$, $j' \in (j)$, $p \in P$, $p' \in P$, $\gamma \in \Gamma$ exist such that

$$(i' + p)\gamma(j' + p') \notin P.$$

Hence

$$i'\gamma(j' + p') - i'\gamma j' + i'\gamma p' + p\gamma(j' + p') \notin P.$$

Hence, since

$$i'\gamma(j' + p') - i'\gamma j' \in P \wedge p\gamma(j' + p') \in P$$

therefore $i'\gamma j' \notin P$. So $I\,\Gamma J \not\subseteq P$, this means **2)** is true.

If $(P_\alpha)_{\alpha \in A}$ is a family of prime ideals of Γ-near-ring $(M, +, \Gamma)$ ordered by inclusion, therefore the intersections:

$$P = \bigcap_{\sim \in A} P_\sim$$

is a prime ideal. To demonstrate what we have been saying until now, initially we line the A group by the equivalence

$$\alpha \le \beta \Leftrightarrow P_\alpha \subseteq P_\beta.$$

It si very clear that P is an ideal of Γ-near-ring M.

Let I, J be two ideals of M such that

$$I \Gamma J \subseteq \bigcap_{\sim \in A} P_\sim \; .$$

So, for every $\alpha \in A$ we have $I \Gamma J \subseteq P_\alpha$. If it exists a $\alpha \in A$ such that $I \not\subseteq P_\alpha$, therefore since P_α is a prime ideal, $J \subseteq P_\alpha$. Hence, for every $\beta \ge \alpha$, $J \subseteq P_\beta$. If it exists a $\lambda < \alpha$ such that

$J \not\subseteq P_\lambda$, therefore since P_λ is a prime ideal, $I \subseteq P_\lambda$ and therefore $I \subseteq P_\alpha$, that is a contradiction. Hence, we have :

$$\forall \alpha \in A, J \subseteq P_\alpha$$

therefore $J \subseteq \bigcap_{\sim \in A} P_\sim$, meaning that $P = J \subseteq \bigcap_{\sim \in A} P_\sim$ is a prime ideal.

Corollary 3.2 *If* d*-near-ring M is a simple, then M is prime or M* d *M = 0.*

Proof. If I, J are two ideals of M, therefore since M is simple we have $I = M$ or $I = 0$ and $J = M$ or $J = 0$. Hence, if we have for the ideals I, J of M the equation $I \Gamma J = 0$, then $I = 0 \vee J = 0$ or $I = J = M$. If $I = 0 \vee J = 0$, then M is prime. If $I = J = M$, then $M \Gamma M = 0$.

Corollary 3.3. *If ideal I of* d*-near-ring M is maximal, then it is prime or M* d *M = I.*

Corollary 3.4. *If (M, +, (ẻ)*d*) is a* d*-near-ring such that for a* ¥ Č d *there is an element which is* ¥*-unit, then every maximal ideal I of M is prime.*

Proof. If for one $\gamma \in \Gamma$ the element e is γ-one of M, then $M\gamma M = \{m_1\gamma m_2 \mid m_1, m_2 \in M\} = M$ because for every $m \in M$, $m = m\gamma e$. Hence, since $M \neq I$ the equation is not true $M \Gamma M = I$. By **the corollary 3.2** the ideal I is prime.

Corollary 3.5. *For every ideal I of* d*-near-ring M exists prim minimal ideal of I.*

Proof. We denote by ***P*** the group of prime ideals of *M* containing *I*. This group is not empty because only *M* is a prime ideal that contains *I*. The intersection of all these prime ideals that contain the ideal *I* is pricesily the minimal prime ideal of ideal *I* because it is included in every prime ideal that contains the ideal *I*. ■

Proposition 3.6. *Let (M, +, (ȇ)*$_d$ *) be a* d *-near-ring zero–symmetric. A subgroup B of group (M, +) is bi-ideal of M in that case and only then B*d*M*d*B* ċ *B* .

<u>Proof</u>. Let B be a bi-ideal of M, that is to say BΓMΓB ∩ (BΓM)Γ∗ B ⊆ B.

Since M is zero – symmetric, for m ∈ M, ¥ ∈ Γ and b ∈ B we have:

m¥ b = m¥ (0 + b) - m¥ 0 ∈ (MΓ) ∗ B that is to say MΓB ⊆ MΓ ∗ B. This way are true the inclusions: BΓMΓB ⊆ (BΓMΓB) ∩ (BΓM) ∗ B ⊆ B and consequently we have

BΓMΓB ⊆ B.

CONCLUSION:

As stated previously, this paper we introduced the concept of prime ideals, maximal ideals and bi – ideals in Γ - near - rings obtaining some characterizations and their links. We also introduced that if M is Γ - near –ring which for a ¥ ∈ M exists an element which is ¥ - unit then every maximal ideal I of M is prime ideal.

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