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String In Topological Vector Space

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Abstract

Introducing the concept of string resulted in a vector space one important step for further Functional Analysis. This made possible the availability of the most important results of this field in other classes spaces. Moreover through the strings was reached to simplified further the validations which became more natural and more elementary.

In this material will be treated quotient space and direct sum of Topological Vector Space as inductive limit. It will also be treated a very important theorem on the relationship between inductive limit with the direct sum and the quotient space.

Introduction

This study is divided in two sections. The first section deliver the general setting of the theory, topological vector spaces definitions and propositions.
In the second section will give the definition of inductive limit, inductive limit examples and the link between inductive limit the direct sum and quotient

1. Strings and linear topologies, topological direct sums, inductive limits.

Definition 1.1

If the vector space (E , + ,*) is a equipped with a topology $\Im$, we denote this topological vector space by (E , $\Im$). $\Im$ is a linear topology if

$S: E\times E \to E$ **dhe** $P: E\times K \to E$

$(x,y) \to x+y$ $(a, x) \to a\,x$

Are continuous the topology $\Im$.

Let E be a vector space over K. A sequence $U = (U_n)_{n \in N}$ of subset U_n of E is colled a string (in E) if

(iv) every $U_n \subset U$ is *balanced*, that means for any $x \in U_n$ and $\lambda \in K, |\lambda| \leq 1$, we have $\lambda x \in U_n$,

(v) every U_n is *absorbing*, that means for any $x \in E$ there is a $\lambda \in K, \lambda > 0$, such that $x \in \lambda U_n$,

(vi) $U = (U_n)_{n \in N}$ is *summative*, that means $U_{n+1} + U_{n+1} \subset U_n$ for all $n \in N$.

Definition 1.2
A topological space (E , $\Im$) is said to be Hausdorff topological space if and only for any pair of distict point $x, y \in E, (x \neq y)$, there exist sets $U, V \in \Im$ such that $x \in U, y \in V$ and $U \cap V = \emptyset$

Definition 1.3
If Φ is linear and Hausdorff , we call (E , $\Im$) a topological vector space (abbreviated: t.v.s)
Let Φ be a linear topology on E. A string U = $(U_n)_{n\in N}$ in (E , $\Im$) is called a topologica string , if every knot U_n is a neighbourhood of 0.
For a string U we denote by the set N(U) = $\bigcap_{n\in N} U_n$. N(U) is called the kernel of U. Since a string U is summative and its knots are balanced, the kernel N(U) is a linear subspace on E.

Definition 1.4
Let Φ be a set of strings in a vector space E, such that for all U, V$\in$ Φ there is a W$\subset$ U$\cap$V. A set Φ is called direct.

Theorem 1.1
The knots of the strings in Φ form a base of 0-neighbourhoods form a linear topology $\Im_\Phi$ on E

Theorem 1.2
A direct set Φ of strings in a vector space E we have (E , $\Im_\Phi$) Hausdorff if only if $\bigcap_{U\in\Phi} N(U)$ = { 0 }.

A directed set Φ of strings in a t.v.s (E , $\Im$) with $\Im = \Im_\Phi$ is called fundamental i.e the knots of the strings in Φ form a base of 0-neighbourhoods in (E , $\Im$) In this case we say: Φ generate $\Im$.

Definition 1.5
Let $(E, \Im)$ be a t.v.s and let Φ be a set of strings in E. Φ is called $\Im$-saturated, if all the elements of Φ are topological and if for any $\Im$-topological string V in $(E, \Im)$ there is an U $\in$ Φ with U$\subset$V .

2. Induktive limit

Let E be a vector space and $(E_i, \Im_i)$ t.v.s. For i∈ I, I an index set, let $A_i : E_i \to E$ be a linear mapping. Assume E = $\sum_{i\in I} A_i(E_i)$ and consider the set Φ of strings in E given by

Φ = { U : U is a string in E and, A_i^{-1} (U) is a topological string in $(E_i, \Im_i)$ for every i∈ I } Then we have.

Theorem 2.1
1.Φ is directed.

2. The topology $\Im_\Phi$ is the finest linear topology on Esuch that allmapings A_i, i∈ I, are continouns.

Note that according to our definition we require an limi $(E, \Im)$ always to be Housdorff. The following theorem gives us a practical way of constructing inductive limit.

Theorem 2.2
If (E,$\Im$) = $\sum_{i\in I} A_i(E_i, \Im_i)$, choose for i∈ I a $\Im_i$-saturated set Φ_i of strings in $(E_i, \Im_i)$. If U $_i$∈ Φ_i, we construct a string U = $(U_n)_{n\in N}$ in E by $U_n = \sum_{k=1}^{\infty} \bigcup_{i\in I} A_i\left(U^i_{2^{n-1}k}\right)$. The set Φ of all these strings U is a saturated set of strings in $(E, \Im) = \sum_{i\in I} A_i(E_i, \Im_i)$

Theorem 2.3
Let (F,$\Im'$) be a t.v.s.. If $(E, \Im) = \sum_{i\in I} A_i(E_i, \Im_i)$ a linear mapping A: (E ,$\Im$) → (F,$\Im'$) is continuous, if and only if $A \circ A_i$ for each $i \in I$.

Examples of inductive limit

Example 2.1 Quotient space as the inducite limit.
Let F be a closed subspace of the t.v.s.$(E, \Im)$. For the quotient space $(E,\Im)/F = \sum_F K_F(E,\Im)$ ku K_F : E → E/F is quotient mapping (K_F (x) = $\dot{x}$ = {x+a : a∈ F})

Example 2.2 Topological direct sum as the inductive limit.

Let $(E_i, \Im_i)_{i\in I}$ be a t.v.s. and denote by E the algebraic direct sum of the E_i, E=$\bigoplus_{i\in I} E_i$

Let $\Im$ be the inductive limit topology on E with respect to the emmbedings
$A_i: (E_i, \Im_i) \to E$ Since E$\subset \prod_{i\in I} E_i$ and since all $I_i : (E_i, \Im_i) \to (E, \Im_{®})$ are continouns
($\Im_{®}$ denotes the topology which is induced by $\prod_{i\in I}(E_i \Im_i)$ on E) we have $\Im_{®} \subset \Im$, and the inductive limit topology on E is Hausdorff. $(E, \Im)$ is called topological direct sum of the $(E_i, \Im_i)_{i\in I}$ and we denote this sum by $(E, \Im) = \left(\bigoplus_{i\in I} E_i, \Im \right) = \sum_{i\in I} A_i(E_i, \Im_i)$.

Example 2.3 Let $\Im_i$,i$\in I$, be a set Hausdorff linear topologies on a vector space E. Let $A_i : (E, \Im_i) \to E$ be the identity mapping. Then the inductive topology $\Im$ on E with respect to $\{(E, \Im_i), A_i\}_{i\in I}$) is the finest linear topology on E which is coarser then all $\Im_i$. $\Im$ is not necessarily Hausdorff.

The following proposition shows that we obtain all inductive limits by forming direct sums and quetionts.
Let $(E_i, \Im_i)_{i\in I}$ be t.v.s. E vector space and $A_i : E_i \to E$ a linear mapping we denote

$F = \bigoplus_{i\in I} E_i$ and define A :F → E such that x ∈ $F = \bigoplus_{i\in I} E_i$ $Ax = A\left(\sum_{i\in I_0} x_i\right) = \sum_{i\in I_0} A_i x_i$

Proposition $(E, \Im) = \sum_{i\in I} A_i(E_i, \Im_i)$, $(F, \Im') = \bigoplus_{i\in I}(E_i, \Im_i)$
A linear mapping $A: (F, \Im') \to (E, \Im)$ is continuous

Theorem 2.4
$(E, \Im)$ is topological isomorphic to the (F/N, $\Im_{F/N}$) where

$$N = \left\{ \sum_{i\in I_0} x_i \quad , \quad \sum_{i\in I} A_i x_i = 0 \right\}$$

3. Conclusion

Starting from the notion of a "string" in a vector space we develop a general theory of topological vector spaces giving most of the results known up to now. They help to develop a theory of topological vector spaces which gives a satisfactory generalization of the inductive limits, topological direst sums and quotients.

4. References

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