On Pure Hyperideals In Ternary Semihypergroups

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ABSTRACT

This paper deals with a class of algebraic hyperstructures called ternary semihypergroups, which are a generalization of ternary semigroups. In this paper we introduce some special classes of hyperideals in ternary semihypergroups. We introduce the notions of pure hyperideals, weakly pure hyperideals, purely and purely prime hyperideals and study some properties of them in ternary semihypergroups and weakly regular ternary semihypergroups. The collection of all proper purely prime hyperideals of a ternary semihypergroup with zero is topologized.

1. INTRODUCTION AND PRELIMINARIES

Algebraic structures play a prominent role in mathematics with wide ranging applications in many disciplines such as theoretical physics, computer sciences, control engineering, information sciences, coding theory etc. Ternary algebraic operations were considered in the 19th century by several mathematicians such as Cayley [1] and later it was generalized by Kapranov, et al. in 1990 [2]. Ternary structures and their generalization, the so-called n-ary structures, raise certain hopes in view of their possible applications in physics and other sciences. Ternary semigroups are universal algebras with one associative operation. The theory of ternary algebraic system was introduced by D. H. Lehmer [3] in 1932. He investigated certain algebraic systems called triplexes which turn out to be commutative ternary groups. The notion of ternary semigroups was introduced by S. Banach (cf. [4]). He showed by an example that a ternary semigroup does not necessary reduce to an ordinary semigroup. In 1965, Sioson [5] studied ideal theory in ternary semigroups. In [6, 7] Dudek et. al. studied the ideals in n-ary semigroups. In 1995, Dixit and Dewan [8] introduced and studied some properties of ideals and quasi-(bi-) ideals in ternary semigroups.

Hyperstructure theory was introduced in 1934, when F. Marty [9] defined hypergroups based on the notion of hyperoperation, began to analyze their properties and applied them to groups. Algebraic hyperstructures are a suitable generalization of classical algebraic structures. In the following decades and nowadays, a number of different hyperstructures are widely studied from the theoretical point of view and for their applications to many subjects of pure and applied mathematics.. In a classical algebraic structure the composition of two elements is an element, while in an algebraic hyperstructure, the composition of two elements is a set.

Ternary semihypergroups are algebraic structures with one associative hyperoperation and they are a particular case of an n-ary semihypergroup (n-semihypergroup) for $n = 3$ (cf. [10-14]). Recently, Hila, Davvaz and et. al. [15-17] introduced and studied some classes of hyperideals in ternary semihypergroups. The main purpose of this paper is to introduce some other special classes of hyperideals in ternary semihypergroups. We introduce the notions of pure hyperideals, weakly pure hyperideals, purely and purely prime hyperideals and study some properties of them in ternary semihypergroups and weakly regular ternary semihypergroups. The collection of all proper purely prime hyperideals of a ternary semihypergroup with zero is topologized.

Recall first the basic terms and definitions from the hyperstructure theory.

Definition 1.1 *A map $\circ : H \times H \to \mathcal{P}^*(H)$ is called hyperoperation or join operation on the set H, where H is a nonempty set and $\mathcal{P}^*(H) = \mathcal{P}(H) \setminus \{\emptyset\}$ denotes the set of all nonempty subsets of H.*

Definition 1.2 *A hyperstructure is called the pair $(H,\circ)$ where $\circ$ is a hyperoperation on the set H.*

Definition 1.3 *A hyperstructure $(H,\circ)$ is called a semihypergroup if for all $x, y, z \in H$, $(x \circ y) \circ z = x \circ (y \circ z)$, which means that $\bigcup_{u \in x \circ y} u \circ z = \bigcup_{v \in y \circ z} x \circ v$.*

If $x \in H$ and ¤ $\neq A, B \subseteq H$, then $A \circ B = \bigcup_{a \in A, b \in B} a \circ b, A \circ x = A \circ \{x\}$ and $x \circ B = \{x\} \circ B$.

Definition 1.4 *A non-empty subset* B *of a semihypergroup* H *is called a sub-semihypergroup of* H *if* $B \circ B \subseteq B$ *and* H *is called in this case super-semihypergroup of B.*

Definition 1.5 *Let* $(H,\circ)$ *be a semihypergroup. Then* H *is called a hypergroup if it satisfies the reproduction axiom, for all* $a \in H$, $a \circ H = H \circ a = H$.

Definition 1.6 *A map* $f : H \times H \times H \to \mathcal{P}^*(H)$ *is called ternary hyperoperation on the set* H, *where* H *is a nonempty set and* $\mathcal{P}^*(H) = \mathcal{P}(H) \setminus \{\varnothing\}$ *denotes the set of all nonempty subsets of* H.

Definition 1.7 *A ternary hypergroupoid is called the pair* (H, f) *where* f *is a ternary hyperoperation on the set* H.

If A, B, C are nonempty subsets of H, then we define

$$f(A,B,C) = \bigcup_{a\in A, b\in B, c\in C} f(a,b,c).$$

Definition 1.8 *A ternary hypergroupoid* (H, f) *is called a ternary semihypergroup if for all* $a_1, a_2, ..., a_5 \in H$, *we have*

$$f(f(a_1,a_2,a_3),a_4,a_5) = f(a_1, f(a_2,a_3,a_4),a_5) = f(a_1,a_2, f(a_3,a_4,a_5)). \quad (*)$$

Since the set $\{x\}$ can be identified with the element x, any ternary semigroup is a ternary semihypergroup. It is clear that due to associative law in ternary semihypergroup (H, f), for any elements $x_1, x_2, ..., x_{2n+1} \in H$ and positive integers m, n with $m \le n$, one may write

$$f(x_1, x_2, ..., x_{2n+1}) = f(x_1, ..., x_m, x_{m+1}, x_{m+2}, ..., x_{2n+1}) =$$
$$f(x_1, ..., f(f(x_m, x_{m+1}, x_{m+2}), x_{m+3}, x_{m+4}), ..., x_{2n+1}).$$

Definition 1.9 *Let* (H, f) *be a ternary semihypergroup. Then* H *is called a ternary hypergroup if* $\forall a,b,c \in H$, $\exists x, y, z \in H$ *such that*

$$c \in f(x,a,b) \cap f(a,y,b) \cap f(a,b,z).$$

Definition 1.10 *Let* (H, f) *be a ternary semihypergroup and* T *a nonempty subset of* H. *Then* T *is called a ternary subsemihypergroup of* H *if and only if*

$f(T,T,T) \subseteq T$.

Definition 1.11 *A ternary semihypergroup* (H, f) *is said to have a zero element if there exist an element* $0 \in H$ *such that for all*

$a,b \in H,\ f(0,a,b) = f(a,0,b) = f(a,b,0) = \{0\}$.

Definition 1.12 *Let* (H, f) *be a ternary semihypergroup. An element* $e \in H$ *is called left identity element of H if for all* $a \in H$, $f(e,a,a) = \{a\}$. *An element* $e \in H$ *is called an identity element of H if for all* $a \in H$,
$f(a,a,e) = f(e,a,a) = f(a,e,a) = \{a\}$. *It is clear that*
$f(e,e,a) = f(e,a,e) = f(a,e,e) = \{a\}$.

Definition 1.13 *A nonempty subset* I *of a ternary semihypergroup* H *is called a left (right, lateral) hyperideal of* H *if*
$f(H,H,I) \subseteq I(f(I,H,H) \subseteq I, f(H,I,H) \subseteq I)$.

A nonemtpy subset I of a ternary semihypergroup H is called a *hyperideal* of H if it is a left, right and lateral hyperideal of H. A nonemtpy subset I of a ternary semihypergroup H is called *two-sided hyperideal* of H if it is a left and right hyperideal of H.

Definition 1.14 *A left hyperideal I of a ternary semihypergroup H is called idempotent if* $f(I,I,I) = I$.

Definition 1.15 *A ternary semihypergroup H is said to be regular if* $\forall a \in H, \exists x \in H, a \in f(a,x,a)$.

It is clear that every ternary hypergroup is a regular ternary semihypergroup.

2. ON PURE HYPERIDEALS IN TERNARY SEMIHYPERGROUPS

In this section we introduce the notions of pure and purely prime hyperideals and we study their properties in ternary semihypergroups and weakly regular ternary semihypergroups.

Definition 2.1 *Let (H, f) be a ternary semihypergroup. A right hyperideal A of H is called a right pure right hyperideal if $\forall x \in A, \exists y, z \in A, x \in f(x, y, z)$. If A is a two-sided hyperideal of H with the property that $\forall x \in A, \exists y, z \in A$, $x \in f(x, y, z)(x \in f(y, z, x))$, then A is called a right(left) pure two-sided hyperideal. If A is a hyperideal of H with the property that $\forall x \in A, \exists y, z \in A$, $x \in f(x, y, z)(x \in f(y, z, x))$, then A is called a right(left) pure hyperideal.*

Left pure left hyperideals are defined analogously.

Example 2.2 *Let $H=\{a,b,c,d,e,g\}$ and $f(x,y,z)=(x*y)*z$, $\forall x, y, z \in H$, where $*$ is defined by the table:*

$*$	a	b	c	d	e	g
a	a	$\{a,b\}$	c	$\{c,d\}$	e	$\{e,g\}$
b	b	b	d	d	g	g
c	c	$\{c,d\}$	c	$\{c,d\}$	c	$\{c,d\}$
d	d	d	d	d	d	d
e	e	$\{e,g\}$	c	$\{c,d\}$	e	$\{e,g\}$
g	g	g	d	d	g	g

Then (H, f) is a ternary semihypergroup. Clearly, $I_1 = \{c, d\}$, $I_2 = \{c, d, e, g\}$ and H are right(left) pure two-sided hyperideals of H.

Example 2.3 *Let $H = \{0,1,x, y, z,t\}$ and $f(x, y, z) = (x \circ y) \circ z$ for all $x, y, z \in H$ with the hyperoperation $\circ$ given by the following table:*

$\circ$	0	x	y	z	t	1
0	0	0	0	0	0	0
x	0	x	$\{x,y\}$	$\{x,z\}$	H	x
y	0	y	y	$\{y,t\}$	$\{y,t\}$	y
z	0	z	$\{z,t\}$	z	$\{z,t\}$	z
t	0	t	t	t	t	t

Then (H, f) is a ternary semihypergroup. It is easy to see that $I_1 = \{0, t\}$, $I_2 = \{0, z, t\}$, $I_3 = \{0, y, t\}$ and $I_4 = \{0, y, z, t\}$ are right pure right hyperideals of *H*.

Example 2.4 *Let* $H = \{a, b, c, d, e, f\}$ *and* $f(x, y, z) = (x \circ y) \circ z$ *for all* $x, y, z \in H$ *with the hyperoperation* $\circ$ *given by the following table:*

$\circ$	0	a	b	c	d	e	f	1
0	0	0	0	0	0	0	0	0
a	0	a	$\{a,b\}$	c	$\{c,d\}$	e	$\{e,f\}$	a
b	0	b	b	d	d	f	f	b
c	0	c	$\{c,d\}$	c	$\{c,d\}$	c	$\{c,d\}$	c
d	0	d	d	d	d	d	d	d
e	0	e	$\{e,f\}$	c	$\{c,d\}$	e	$\{e,f\}$	e
f	0	f	f	d	d	f	f	f
1	0	a	b	c	d	e	f	1

Then (H, f) is a ternary semihypergroup. Clearly, $I_1 = \{0, d\}$, $I_2 = \{0, d, f\}$ and $I_3 = \{0, b, d, f\}$ are right pure right hyperideals of *H*. $I_4 = \{0, c, d\}$ is a two-sided hyperideal of H which is a right pure hyperideal but not a left pure hyperideal. $I_5 = \{0, c, d, e, f\}$ is a two-sided hyperideal of H which is a right and left pure hyperideal.

Proposition 2.5 *Let* (H, f) *be a ternary semihypergroup. Let* A *be a two-sided hyperideal of* H. *Then* A *is right pure if and only if for any right hyperideal* B, $B \cap A = f(B, A, A)$.

Proof. Suppose A is a right pure two-sided hyperideal of H. For every right hyperideal B of H, we have always $f(B, A, A) \subseteq B \cap A$. Let $x \in B \cap A$. Since A is a right pure two-sided hyperideal, there exist $y, z \in A$ such that $x \in f(x, y, z)$. As $x \in B$ and $y, z \in A$, $x \in f(x, y, z) \subseteq f(B, A, A)$. Hence $x \in f(B, A, A)$. This implies that $B \cap A \subseteq f(B, A, A)$. Thus $B \cap A = f(B, A, A)$. *Conversely,* assume $B \cap A = f(B, A, A)$, for any right hyperideal B of H. We show that A is a right pure two-sided hyperideal. Let $x \in A$ and $B = \{x\} \cup f(x, H, H)$ be the right hyperideal of H generated by x. Then we have

$$(\{x\} \cup f(x, H, H)) \cap A = f((\{x\} \cup f(x, H, H)), A, A) = f(x, A, A) \cup f(f(x, H, H), A, A) \subseteq$$

$\subseteq f(x,A,A) \cup f(x,A,A) = f(x,A,A)$.

Since $x \in (\{x\} \cup f(x,H,H)) \cap A$, we have $x \in f(x,A,A)$. Hence there exist $y,z \in A$ such that $x \in f(x,y,z)$. Thus A is right pure.

Similarly, it can be shown that a hyperideal A of a ternary semihypergroup H is right pure if and only if $B \cap A = f(B,A,A)$ for every right hyperideals B of H.

Definition 2.6 *A ternary semihypergroup (H, f) is said to be right weakly regular if for each $x \in H, x \in f(f(x,H,H), f(x,H,H), f(x,H,H))$.*

It is clear that every regular ternary semihypergroup is right weakly regular but the converse is not true.

Theorem 2.7 *Let (H, f) be a ternary semihypergroup. The following statements are equivalent:*

1. H *is right weakly regular*.

2. *Every right hyperideal of H is idempotent, i.e. $B=f(B,B,B)$ for every right hyperideal B of H.*

3. $B \cap A = f(B,A,A)$ *for every right hyperideal B and two-sided hyperideal A of H.*

4. $B \cap A = f(B,A,A)$ *for every right hyperideal B and for every hyperideal A of H.*

Proof. $(1) \Rightarrow (2)$. Let B be a right hyperideal of H, then $f(B,B,B) \subseteq f(B,H,H) \subseteq B$. Let $x \in B$. Then $x \in f(f(x,H,H), f(x,H,H), f(x,H,H)) \subseteq f(B,B,B)$. Thus $B \subseteq f(B,B,B)$. Hence $B = f(B,B,B)$.

$(2) \Rightarrow (1)$. Let us suppose that every right hyperideal of H is idempotent. Let $x \in H$. Then $B = \{x\} \cup f(x,H,H)$ is the right hyperideal of H, so idempotent, i.e.

$$\{x\}\cup f(x,H,H) = f((\{x\}\cup f(x,H,H)),(\{x\}\cup f(x,H,H)),(\{x\}\cup f(x,H,H)))$$

$$= f(x,x,x)\cup f(f(x,x,x),H,H)\cup f(x,x,H,H,x)\cup f(x,x,H,H,x,H,H)\cup f(x,H,H,x,x)\cup$$

$$\cup f(x,H,H,x,x,H,H)\cup f(x,H,H,x,H,H,x)\cup f(x,H,H,x,H,H,x,H,H).$$

Doing simple calculation, we have $x\in f(f(x,H,H),f(x,H,H),f(x,H,H))$. Hence H is right weakly regular.

$(1)\Rightarrow(3)$. Let us suppose that H is right weakly regular ternary semihypergroup and B is a right hyperideal and A a two-sided hyperideal of H. Then $f(B,A,A)\subseteq B\cap A$. Let $x\in B\cap A$. Since H is right weakly regular, we have $x\in f(f(x,H,H),f(x,H,H),f(x,H,H))$. Hence $x\in f(B,A,A)$, which shows that $B\cap A\subseteq f(B,A,A)$. Hence $B\cap A = f(B,A,A)$.

$(3)\Rightarrow(4)$. It is obvious.

$(4)\Rightarrow(1)$. Let $x\in H$ and $B=\{x\}\cup f(x,H,H)$ be the right hyperideal of H generated by x,
$A=\{x\}\cup f(x,H,H)\cup f(H,H,x)\cup f(H,x,H)\cup f(H,H,x,H,H)$ be the hyperideal of H generated by x. Then we have,

$$(\{x\}\cup f(x,H,H))\cap(\{x\}\cup f(x,H,H)\cup f(H,H,x)\cup f(H,x,H)\cup f(H,H,x,H,H) =$$

$$= f((\{x\}\cup f(x,H,H)),(\{x\}\cup f(x,H,H)\cup f(H,H,x)\cup f(H,x,H)\cup f(H,H,x,H,H)),$$
$$(\{x\}\cup f(x,H,H)\cup f(H,H,x)\cup$$
$$f(H,x,H)\cup f(H,H,x,H,H)))=(f(x,x,x)\cup f(x,x,x,H,H)\cup$$

$$\cup f(x,x,H,H,x)\cup f(x,x,H,x,H)\cup f(x,x,H,H,x,H,H)\cup f(x,H,H,x,x)\cup$$

$$\cup f(x,H,H,x,x,H,H)\cup f(x,H,H,x,H,H,x)\cup f(x,H,H,x,H,x,H)\cup$$

$\cup f(x,H,H,x,H,H,x,H,H) \cup f(x,H,x,H,x) \cup f(x,H,x,H,x,H,H) \cup f(x,H,x,H,H,x,H).$

By simple calculations, we have $x \in f(f(x,H,H), f(x,H,H), f(x,H,H))$. Hence H is right weakly regular ternary semihypergroup.

Theorem 2.8 *Let* (H, f) *be a ternary semihypergroup. The following statements are equivalent:*

1. H *is right weakly regular*.
2. *Every two-sided hyperideal* A *of* H *is right pure.*
3. *Every hyperideal* A *of* H *is right pure.*

Proof. It follows by Theorem 2.7 and Proposition 2.5.

Proposition 2.9 *Let* (*H,f*) *be a ternary semihypergroup with 0. The following statements hold true:*

1. $\{0\}$ *is a right pure hyperideal of* H.
2. *Any union of any number of right pure two-sided hyperideals (hyperideals) of* H *is a right pure two-sided hyperideal (hyperideal) of* H.
3. *Any finite intersection of right pure two-sided hyperideals (hyperideals) of* H *is a right pure two-sided hyperideal (hyperideal) of* H.

Proof. (1). $\{0\}$ is obviously right pure hyperideal of H.

(2). Let $\{I_k\}_{k\in K}$ be a family of right pure two-sided hyperideals of H. Then $\bigcup_{k\in K} I_k$ is a two-sided hyperideal of H. Let us suppose that $x \in \bigcup_{k\in K} I_k$. Then $\exists k \in K$ such that $x \in I_k$. Since I_k is a right pure two-sided hyperideal of H, there exist $y, z \in I_k$ such that $x \in f(x, y, z)$. It follows that $y, z \in \bigcup_{k\in K} I_k$ such that $x \in f(x, y, z)$. Hence $\bigcup_{k\in K} I_k$ is a right pure two-sided hyperideal of H.

(3). Let I_1 and I_2 be right pure hyperideals of H and let $x \in I_1 \cap I_2$. Since $x \in I_1, x \in I_2$ and I_1, I_2 are right pure two-sided hyperideals of H, there exist $y_1, z_1 \in I_1$ and $y_2, z_2 \in I_2$ such that $x \in f(x, y_1, z_1$ and $x \in f(x, y_2, z_2)$. Thus we have $x \in f(x, y_1, z_1) \subseteq f(f(x, y_2, z_2), y_1, z_1)$
$\subseteq f(f(f(x, y_1, z_1), y_2, z_2), y_1, z_1) \subseteq f(x, f(y_1, z_1, y_2), f(z_2, y_1, z_1))$, where $f(y_1, z_1, y_2), f(z_2, y_1, z_1) \subseteq I_1 \cap I_2$. Thus $I_1 \cap I_2$ is a right pure hyperideal of H.

Similarly it can be proved the case of hyperideal.

Proposition 2.10 *Let (H, f) be a ternary semihypergroup with 0 and A be any two-sided hyperideal of H. Then A contains a largest right pure two-sided hyperideal (It is called the pure part of A and denote by $\mathcal{S}(A)$).*

Proof. Let $\mathcal{S}(A)$ be the union of all right pure two-sided hyperideals contained in A. Such hyperideals exist because $\{0\}$ is a right pure hyperideal contained in each two-sided hyperideal. By the above proposition $\mathcal{S}(A)$ is a right pure two-sided hyperideal. It is indeed the largest right pure two-sided hyperideal contained in A.

Proposition 2.11 *Let (H, f) be a ternary semihypergroup with 0. Let A, K be two-sided hyperideals of H and $\{A_i\}_{i \in I}$ be the family of two-sided hyperideals of H. Then*

1. $\mathcal{S}(A \cap K) = \mathcal{S}(A) \cap \mathcal{S}(K)$.
2. $\bigcup_{i \in I} \mathcal{S}(A_i) \subseteq \mathcal{S}(\bigcup_{i \in I} A_i)$.

Proof. (1). By $\mathcal{S}(A) \subseteq A, \mathcal{S}(K) \subseteq K$ it follows $\mathcal{S}(A) \cap \mathcal{S}(K) \subseteq A \cap K$. But $\mathcal{S}(A) \cap \mathcal{S}(K)$ is right pure by Proposition 2.9, so $\mathcal{S}(A) \cap \mathcal{S}(K) \subseteq \mathcal{S}(A \cap K)$. On the other hand $\mathcal{S}(A \cap K) \subseteq \mathcal{S}(A)$. Similarly, $\mathcal{S}(A \cap K) \subseteq \cap \mathcal{S}(K)$. Thus $\mathcal{S}(A \cap K) \subseteq \mathcal{S}(A) \cap \mathcal{S}(K)$. Hence, $\mathcal{S}(A \cap K) = \mathcal{S}(A) \cap \mathcal{S}(K)$.

(2). By $\mathcal{S}(A_i) \subseteq A_i$ it follows $\bigcup_{i \in I} \mathcal{S}(A_i) \subseteq \bigcup_{i \in I} A_i$. Since $\mathcal{S}(A_i)$ is right pure, we have $\bigcup_{i \in I} \mathcal{S}(A_i)$ is right pure. Thus we have $\bigcup_{i \in I} \mathcal{S}(A_i) \subseteq \mathcal{S}(\bigcup_{i \in I} A_i)$.

Definition 2.12 *Let (H, f) be a ternary semihypergroup and A be a right pure two-sided hyperideal of H. Then A is called purely maximal if A is maximal in the lattice of proper right pure two-sided hyperideals of H.*

A proper right pure two-sided hyperideal A of H is called *purely prime* if $f(A_1, H, A_2) \subseteq A$ implies $A_1 \subseteq A$ or $A_2 \subseteq A$ for any right pure two-sided hyperideals A_1, A_2 of H. Equivalently, $A_1 \cap A_2 \subseteq A$ implies $A_1 \subseteq A$ or $A_2 \subseteq A$. This is because $f(A_1, H, A_2) \subseteq A_1 \cap A_2$ and $A_1 \cap A_2 = f(A_1, A_2, A_2) \subseteq f(A_1, H, A_2)$. Thus $f(A_1, H, A_2) = A_1 \cap A_2$.

Proposition 2.13 *Let (H, f) be a ternary semihypergroup. Then any purely maximal two-sided hyperideal is purely prime.*

Proof. Let us suppose that A is purely maximal two-sided hyperideal of H and A_1, A_2 are right pure two-sided hyperideals of H such that $A_1 \cap A_2 \subseteq A$. Let us suppose that $A_1 \nsubseteq A$. Then $A_1 \cup A$ is a right pure hyperideal such that $A \subsetneq A_1 \cup A$. Since A is purely maximal, so $A_1 \cup A = H$. Thus

$A_2 = A_2 \cap H = A_2 \cap (A_1 \cup A) = (A_2 \cap A_1) \cup (A_2 \cap A) \subseteq A \cup A = A$. Hence A is purely prime.

Proposition 2.14 *Let (H, f) be a ternary semihypergroup with 0. Then the pure part of any maximal two-sided hyperideal of H is purely prime.*

Proof. Let M be a maximal two-sided hyperideal of H and $\mathcal{S}(M)$ be its pure part. Let us suppose $A_1 \cap A_2 \subseteq \mathcal{S}(M)$ where A_1, A_2 are right pure two-sided hyperideals of H. If $A_1 \subseteq M$, then $A_1 \subseteq \mathcal{S}(M)$. If $A_1 \nsubseteq \mathcal{S}(M)$, then $A_1 \nsubseteq M$. Thus $A_1 \cup M = H$ since M is maximal. Hence we have $A_2 = A_2 \cap H = A_2 \cap (A_1 \cup M) = (A_2 \cap A_1) \cup (A_2 \cap M) \subseteq$
$\subseteq \mathcal{S}(M) \cup M \subseteq M \cup M = M$. But $\mathcal{S}(M)$ is the largest right pure two-sided hyperideal contained in M. Thus $A_2 \subseteq \mathcal{S}(M)$. Hence $\mathcal{S}(M)$ is purely prime.

Proposition 2.15 *Let (H, f) be a ternary semihypergroup. Let A be a right pure two-sided hyperideal of H and $a \in H$ such that $a \notin A$, then there exists a purely prime two-sided hyperideal B of H such that $A \subseteq B$ and $a \notin B$.*

Proof. Let
$X = \{B : B \text{ is a right pure two-sided hyperideal of } H, A \subseteq B \text{ and } a \notin B\}$.
Since $A \in X$, then $X \neq \emptyset$. Further, X is partially ordered by inclusion. Let $\{B_i\}_{i\in I}$ be any totally ordered subset of X. By Proposition 2.9, $\bigcup_{i\in I} B_i$ is a right pure two-sided hyperideal. Since $B \subseteq \bigcup_{i\in I} B_i$ and $a \notin \bigcup_{i\in I} B_i$, so $\bigcup_{i\in I} B_i \in X$. Thus by Zorn's Lemma, X has a maximal element, let it be denoted by B, such that B is pure, $A \subseteq B$ and $a \notin B$. Let we prove that B is purely prime. Let us suppose that A_1 and A_2 are right pure two-sided hyperideals of H such that $A_1 \not\subseteq B$ and $A_2 \not\subseteq B$. Since A_1, A_2 and B are right pure, then $A_i \cup B$ is a right pure two-sided hyperideal such that $B \subsetneq A_i \cup B$. Thus $a \in A_i \cup B (i = 1,2)$. As $a \notin B$, we have $a \in A_i (i = 1,2)$. Thus $a \in A_1 \cap A_2$. Hence $A_1 \cap A_2 \not\subseteq B$. This shows that B is purely prime.

Proposition 2.16 *Let (H, f) be a ternary semihypergroup. Then every proper right pure two-sided hyperideal A of H is the intersection of all the purely prime two-sided hyperideals of H containing A.*

Proof. By Proposition 2.15, there exists purely prime two-sided hyperideals containing A. Let $\{B_i\}_{i\in I}$ be the family of all purely prime two-sided hyperideals of H which contains A. Since $A \subseteq B_i$ for all $i \in I$, then $A \subseteq \bigcap_{i\in I} B_i$. Let we show now that $\bigcap_{i\in I} B_i \subseteq A$. Let $a \notin A$. Then by Proposition 2.15, there exists a purely prime two-sided hyperideal B such that $A \subseteq B$ and $a \notin B$. it follows that $a \notin \bigcap_{i\in I} B_i$. Thus $\bigcap_{i\in I} B_i \subseteq A$. Hence $A = \bigcap_{i\in I} B_i$.

3. ON WEAKLY PURE HYPERIDEALS IN TERNARY SEMIHYPERGROUPS

In this section the notion of weakly pure hyperideal of ternary semihypergroups is introduced as a generalization of the pure two-sided hyperideal.

Definition 3.1 *Let (H, f) be a ternary semihypergroup. A two-sided hyperideal A of H is called left (resp. right) weakly pure if $A \cap B = f(A, A, B)$ (resp. $A \cap B = f(B, A, A)$) for all two-sided hyperideals of H.*

It is clear that every left (right) pure two-sided hyperideal is left (right) weakly pure.

Proposition 3.2 *Let (H, f) be a ternary semihypergroup with 0 and A, B be two-sided hyperideals of H. Then*

$$BA^{-1} = \{h \in H : f(x, y, h) \subseteq B, \forall x, y \in A\} \text{ and}$$
$$A_{-1}B = \{h \in H : f(h, x, y) \subseteq B, \forall x, y \in A\}$$

are two-sided hyperideals of H.

Proof. We have $BA^{-1} \neq \emptyset$ because $0 \in BA^{-1}$. Let $r, s \in H$ and $h \in BA^{-1}$. Then for all $x, y \in A, f(x, y, f(s, r, h)) = f(x, f(y, s, r), h) \subseteq B$ because $f(y, s, r) \subseteq A$. Hence $f(s, r, h) \subseteq BA^{-1}$. Also, $f(x, y, f(h, s, r)) = f(f(x, y, h), s, r) \subseteq f(B, H, H) \subseteq B$, because $f(x, y, h) \subseteq B$. Thus $f(h, s, r) \subseteq BA^{-1}$. Hence BA^{-1} is a two-sided hyperideal of H. Let now $s, r \in H$ and $h \in A_{-1}B$. Then $f(f(s, r, h), x, y) = f(s, r, f(h, x, y)) \subseteq f(H, H, B) \subseteq B$ for all $x, y \in A$, because $f(h, x, y) \subseteq B$. Hence $f(s, r, h) \subseteq A_{-1}B$. Also, $f(f(h, s, r), x, y) = f(h, f(s, r, x), y) = f(h, x, y) \subseteq B$, because $f(s, r, x) \subseteq A$. Thus $f(h, s, r) \subseteq A_{-1}B$. Hence $A_{-1}B$ is a two-sided hyperideal of H.

Proposition 3.3 *Let (H, f) be a ternary semihypergroup and A be a two-sided hyperideal of H. Then the following statements are equivalent:*

1. *A is left (right) weakly pure.*
2. $(BA^{-1}) \cap A = B \cap A \, (A_{-1}B \cap A = A \cap B)$ *for all hyperideals B of H.*

Proof. $(1) \Rightarrow (2)$. Let us suppose that A is weakly pure. Since BA^{-1} is a two-sided hyperideal, we have $(BA^{-1}) \cap A = f(A, A, BA^{-1})$. Let we show now that

$f(A,A,BA^{-1}) \subseteq B$. Let $f(a,h,x) \subseteq f(A,A,BA^{-1})$, where $a,h \in A, x \in BA^{-1}$. Then $f(a,h,x) \subseteq B$ by the definition. Hence $f(A,A,BA^{-1}) \subseteq B$. Also $f(A,A,BA^{-1}) \subseteq f(A,A,H) \subseteq A$ and $BA^{-1} \cap A = f(A,A,BA^{-1}) \subseteq A \cap B$. Thus $(BA^{-1}) \cap A \subseteq B \cap A$. Let $b \in B \cap A$, then $f(x,y,b) \subseteq B$ for all $x,y \in A$. Hence $b \in BA^{-1}$. Thus $B \cap A \subseteq (BA^{-1}) \cap A$. Therefore $(BA^{-1}) \cap A = B \cap A$.

$(2) \Rightarrow (1)$. Let us assume that A,B are two-sided hyperideals of a ternary semihypergroup H and $(BA^{-1}) \cap A = B \cap A$. We show that A is left weakly pure. First we show that $B \subseteq f(A,A,B)A^{-1}$. Let $b \in B$, then for every $x,y \in A$, we have $f(x,y,b) \subseteq f(A,A,B)$. Thus $b \in f(A,A,B)A^{-1}$. This shows that $B \subseteq f(A,A,B)A^{-1}$. Thus

$A \cap B \subseteq f(A,A,B)A^{-1} \cap A = f(A,A,B) \cap A \subseteq f(A,A,B)$ by hypothesis. But we have always $f(A,A,B) \subseteq A \cap B$. Hence $A \cap B = f(A,A,B)$. Thus A is left weakly pure.

Proposition 3.4 *Let (H,f) be a ternary semihypergroup. Then the following statements are equivalent:*

1. *Every two-sided hyperideal of H is left weakly pure.*
2. *Every two-sided hyperideal of H is idempotent.*
3. *Every two-sided hyperideal of H is right weakly pure.*

Proof. $(1) \Rightarrow (2)$. Let us suppose that every two-sided hyperideal of H is left weakly pure. Let X be a two-sided hyperideal of H, then for every two-sided hyperideal of H, we have $X \cap Y = f(X,X,Y)$. In particular, $X = X \cap X = f(X,X,X)$. Hence every two-sided hyperideal of H is idempotent.

$(2) \Rightarrow (1)$. Let us suppose that every two-sided hyperideal of H is idempotent. Let X be a two-sided hyperideal of H, then for any two-sided hyperideal Y of H, we have $f(X,X,Y) \subseteq X \cap Y$. On the other hand, we have $X \cap Y = f(X \cap Y, X \cap Y, X \cap Y) \subseteq f(X,X,Y)$.

Hence we have $X \cap Y = f(X,X,Y)$. Thus X is left weakly pure.

$(2) \Rightarrow (3)$. It is proved in similar way as $(2) \Rightarrow (1)$.

$(3) \Rightarrow (2)$. Let us suppose that every two-sided hyperideal of H is right weakly pure. Let X be any two-sided hyperideal of H. Then X is right weakly pure. Hence for every two-sided hyperideal Y of H, we have $X \cap Y = f(Y, X, X)$. In particular $X \cap X = f(X, X, X)$. Hence every two-sided hyperideal of H is idempotent.

4. ON PURE HYPERRADICAL OF A TERNARY SEMIHYPERGROUP

In this section we deal with (H, f) to be a ternary semihypergroup with 0 such that $f(H,H,H) = H$. Let $\mathcal{RP}(H)$ be the set of all right pure hyperideals of H and $\mathcal{PP}(H)$ be the set of all proper purely prime hyperideals of H. Define for every $I \in \mathcal{RP}(H)$,

$$\mathcal{B}_I = \{J \in \mathcal{PP}(H) : I \not\subseteq J\},\ \Im(H) = \{\mathcal{B}_I : I \in \mathcal{RP}(H)\}.$$

Theorem 4.1 $\Im(H)$ *forms a topology on* $\mathcal{PP}(H)$.

Proof. Since $\{0\}$ is a right pure hyperideal of H, then $\mathcal{B}_{\{0\}} = \{J \in \mathcal{PP}(H) : \{0\} \not\subseteq J\} = \varnothing$, because 0 belongs to every right pure hyperideal. Since H is a right pure hyperideal of H, $\mathcal{B}_H = \{J \in \mathcal{PP}(H) : H \not\subseteq J\} = \mathcal{PP}(H)$ because $\mathcal{PP}(H)$ is the set of all proper purely prime hyperideals of H. Let $\mathcal{B}_{I_\sim} : \sim \in \Lambda\} \subseteq \Im(H)$, then

$$\bigcup_{\sim \in \Lambda} \mathcal{B}_{I_\sim} = \{J \in \mathcal{PP}(H) : I_\sim \not\subseteq J \text{ for some } \sim \in \Lambda\} =$$
$$\{J \in \mathcal{PP}(H) : \cup I_\sim \not\subseteq J\} = \mathcal{B}_{\cup I_\sim}.$$

To prove that $\mathcal{B}_{I_1} \cap \mathcal{B}_{I_2} \in \Im(H)$ for every $\mathcal{B}_{I_1}, \mathcal{B}_{I_2} \in \Im(H)$, we consider $J \in \mathcal{B}_{I_1} \cap \mathcal{B}_{I_2}$. Then $J \in \mathcal{PP}(H)$, $I_1 \not\subseteq J$ and $I_2 \not\subseteq J$. Let us suppose that $I_1 \cap I_2 \subseteq J$. Since J is a purely prime hyperideal, therefore either $I_1 \subseteq J$ or $I_2 \subseteq J$, which is a contradiction. Hence $I_1 \cap I_2 \not\subseteq J$, which implies $J \in \mathcal{B}_{I_1 \cap I_2}$. Thus $\mathcal{B}_{I_1} \cap \mathcal{B}_{I_2} \subseteq \mathcal{B}_{I_1 \cap I_2}$. On the other hand, if $J \in \mathcal{B}_{I_1 \cap I_2}$, then $I_1 \cap I_2 \not\subseteq J \Rightarrow I_1 \not\subseteq J$

and $I_2 \not\subseteq J \Rightarrow J \in \mathcal{B}_{I_1}$ and $J \in \mathcal{B}_{I_2} \Rightarrow J \in \mathcal{B}_{I_1} \cap \mathcal{B}_{I_2}$. Hence $\mathcal{B}_{I_1 \cap I_2} \subseteq \mathcal{B}_{I_1} \cap \mathcal{B}_{I_2}$. Consequently, $\mathcal{B}_{I_1 \cap I_2} = \mathcal{B}_{I_1} \cap \mathcal{B}_{I_2}$, which implies $\mathcal{B}_{I_1} \cap \mathcal{B}_{I_2} \in \Im(H)$. Thus $\Im(H)$ is a topology on $\mathcal{PP}(H)$.

CONCLUSION

In this paper we studied a class of algebraic hyperstructures called ternary semihypergroups which are a generalization of ternary semigroups. The main purpose of this paper was to introduce some special classes of hyperideals in ternary semihypergroups. We introduced the notions of pure hyperideals, weakly pure hyperideals, purely and purely prime hyperideals and studied some properties of them in ternary semihypergroups and weakly regular ternary semihypergroups. The collection of all proper purely prime hyperideals of a ternary semihypergroup with zero was topologized.

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