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Consideration On Portfolio Optimization. Beyond The Markowitz Model.

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ABSTRACT.

The paper presents some consideration on consideration on portfolio investment: beyond the Markowitz model. Its matter contains the following issues:

-*Introduction*

-*A view on portfolio optimization*

-*An example of deterministic Markowitz models*

-*The model solution via interior – point Method.*

-*Beyond the Markowitz model.*

-*Conclusion*

It is underlined the portfolio optimization is importante in the optimization of investment in our country and there are the possibility to generalise the deterministic models in the sense of the stochastic optimisation theory [3], beyond the Markowitz models..

Key words: P*ortfolio, Markowitz model, optimization, Lagrangian- multiplier, interior- point method , stochastic , investment, C++, program.*

1. INTRODUCTION.

Harry Markowitz received the 1990 Nobel Prize in economics for his portfolio optimization model in which the tradeoff between risk and reward is explicitly treated. First we shall describe this model in its simplest form [14].

As it is known a portfolio is a collection of financial assets consisting of investments tools as stocs , bonds, gold, foreign exchange, asset-backed securities real estate and bank deposits which are held by a person or a group of persons[12]

Given a collection of potential investment [9] (indexed , say from 1 to n), let R_j denote the return in the next time period on investment j , j= 1,2,…..n. In general R_j is a random variable , although some investment may be essentially deterministic.

Markowitz defined the risk as the variance D of the return :

$$\operatorname{var}(R) = \sum_j x_j \left(R_j - E(R_j)\right)^2 = E \sum_j x_j \overline{R}_j^2 \qquad (1)$$

where $\overline{R}_j = R_j - E\left(R_j\right)$.

One would like to maximize the reward while at the same time not incurring excessive risk. The Markowitz Model[14] *is a quadratic programming one* :

$$\begin{aligned} &\min - \sum x_j E(R_j) + « E \left(\sum x_j \overline{R}_j\right)^2 \\ &\sum_j x_j = 1 \qquad (2) \\ &x \geq 0 \qquad j = 1,2.........n \end{aligned}$$

Here µ is a positive parameter that represents the importance of risk relative to reward..

Below we will use the following notations(symbols):

n – number of assets

C_0– capital that can be invested in euros.

C_{end} - capital at the end of the period in euros.
Q_{EF} - allocation when when the portfolio is on the efficient frontier
R_p- total portfolio return
m_p - expected portfolio return , in euros
s_p- variance portfolio return
r_i- rate of return on asset i
m_i- expected rate of return on asset i.
r_{ij} – correlation between asset I and asset j.
s_{ij} – covariance of asset i and j. $_p$
θ_i = amount invested in asset i, in euros.
μ_f =rate of return on the risk-free asset.
R_f = total return on the risk-free asset.

γ = parameter of absolute risk aversion.
s = slope of the capital market line [1].

$$\sum = \begin{pmatrix} \pm_{11} & \pm_{12} & \cdots & \pm_{1n} \\ \pm_{21} & \ddots & & \vdots \\ \vdots & & & \\ \pm_{n1} & \cdots & & \pm_{nn} \end{pmatrix} = \textit{matrix of covariances of } r.$$

2. A VIEW ON PORTFOLIO OPTIMIZATION.

As it is known the efficient frontier is the curve that shows all efficient portfolios in a risk –return framework. An efficient portfolio maximizes the expected return for a given amount of risk(sometimes standard deviation) or the portfolio that minimizes the risk subject to a given expected return.

The objective function which minimizes the risk is :

$$var(C_{end}) = var(C_0 + R_p) = var(R_p) = var(r^T\theta) = \theta^T\Sigma\theta.$$

(3)

On these circumstances , we have to resolve the problem of the investment policy of minimum variance [1]:

$$Min\{\ \theta^T\Sigma\theta \mid A^T\theta = B\ \}$$

(4)

with

$$A = (\ \mu \quad \bar{1}\ ) \qquad \text{and} \qquad B = \begin{pmatrix} \mu_p \\ C_0 \end{pmatrix}$$

(4_1)

Noting

$$a = \mu^T\Sigma^{-1}\mu,$$
$$b = \mu^T\Sigma^{-1}\bar{1} = \bar{1}^T\Sigma^{-1}\mu,$$
$$c = 1^T\Sigma^{-1}1.$$
$$d = ac - b^2$$

(4_2)

Solving the system we find

$$\theta_{EF} - \Sigma^{-1}A\lambda - \Sigma^{-1}AH^{-1}B - \frac{c\mu_p - bC_0}{d}\Sigma^{-1}\mu + \frac{aC_0 - b\mu_p}{d}\Sigma^{-1}\bar{1}$$

$$= \frac{1}{d}\Sigma^{-1}\left((a\bar{1} - b\mu)C_0 + (c\mu - b\bar{1})\mu_p\right) \qquad (5)$$

2.1 MINIMUM VARIANCE PORTFOLIO

The minimum variance portfolio can be calculated by minimizing the variance , subject to the necessary constraint that un investor can only invest the amount of the capital he has The minimization problem is: [1]

$$Min\left\{ {}^{-T}\sum{}^{-} \;\middle|\; \bar{1}^{T-} = C_0 \right\} \qquad (6)$$

We find the minimum variance and the expected return

$$\pm^2_{mv} = {}^{-T}\sum{}^{-} = \frac{C_0}{c}\left(\sum{}^{-1}\bar{1}\right)^T \sum \frac{C_0}{c}\sum{}^{-1}\bar{1} = \left(\frac{C_0}{c}\right)^2 \bar{1}^T\left(\sum{}^{-1}\right)^T \sum\sum{}^{-1}\bar{1}$$

$$= \left(\frac{C_0}{c}\right)^2 \bar{1}^T \sum{}^{-1}\bar{1} = \left(\frac{C_0}{c}\right)^2 c = \frac{C_0^2}{c} \qquad (7)\ (8)$$

$$«_{mv} = «^{T-} = «^T \sum{}^{-1}\bar{1}\frac{C_0}{c} = b\frac{C_0}{c} = \frac{b}{c}C_0$$

2.2 TANGENCY PORTOFOLIO.

Another preference of investor is the portfolio with maximum Sharpe ratio which is defined a as the return – risk ratio, so [5]

$$\text{Sharpe ratio} = \frac{\text{mean}}{\text{standard deviation}} \qquad (9)$$

Graphically , the portfolio with the maximum Sharpe ratio is the point where a line through the origin is tangent to the efficient frontier ,At the tangency point the corresponding s_{tg} is :

$$\pm_{tg} = \sqrt{\frac{1}{d}\left(c\frac{a^2}{c^2}C_0^2 - \frac{2ab}{b}C_0^2 + aC_0^2\right)} = \frac{\sqrt{a}}{b}C_0 \qquad (10)$$

$$-_{tg} = \frac{c\frac{a}{b}C_0 - bC_0}{d}\sum^{-1}\text{«} + \frac{aC_0 - b\frac{a}{b}C_0}{d}\sum^{-1}\bar{1} = \sum^{-1}\text{«}\frac{C_0}{b} \qquad (11)$$

2.3. OPTIMAL PORTOFOLIO

The theory of Markowitz says the investors goal is to maximize his utility function u :

$$u = E(C_{end}) - \frac{1}{2}\text{¥}\ \mathrm{var}(C_{end}) \qquad (12)$$

In order to calculate the optimal portfolio , we have to maximize the utility subject to the budget constraint[1]

$$Max\left\{C_0 + \text{«}^{T}- - \frac{1}{2}\text{¥}-^{T}\sum- \ |\bar{1}^{T}- = C_0\right\} \qquad (13)$$

Solving this problem we obtain :

$$\text{«}_{opt} == \frac{a}{\text{¥}} + \frac{b}{c}\left(C_0 - \frac{b}{\text{¥}}\right) = \frac{ac - b^2}{c\text{¥}} + \frac{b}{c}C_0 = \frac{d}{c\text{¥}} + \text{«}_{mv} \qquad (14)$$

$$\pm^2_{opt} = -^{T}\sum- = \frac{ac - b^2 + \text{¥}^2 C_0^2}{c\text{¥}^2} = \frac{d}{c\text{¥}^2} + \pm^2_{mv} \qquad (15)$$

2.4.PORTFOLIO WITH A RISK- FREE ASSET.

The theory of Markowitz learns that the new efficient frontier is a stright line , starting at the free risk point and tangent to the old efficient frontier called the Capital Market line.. Because the risk free asset is uncorrelated with free risk assets we have the following relation- ship

$$\pm^2_p = -^{T}\sum- \qquad and \qquad \text{«}_p = \text{«}^{T}- + \text{«}_f -_f \qquad (16)$$

The budget constraint change is

$$\bar{1}^{T}- + -_f = C_0 \qquad (17)$$

and the problem of maximization [1] of the expected return given some variance is:

$$Max\left\{ «^{T}\bar{} + «_{f}\bar{}_{f} \;\middle|\; \begin{array}{l} \bar{1}\bar{} + \bar{}_{f} = C_0 \\ ±_{p}^{2} = \bar{}^{T}\sum\bar{} \end{array} \right\} \tag{18}$$

The expected portofolio return is :

$$\mu_p = \left(\sqrt{c\mu_f^2 - 2b\mu_f + a}\right)\sigma_p + \mu_f C_0 \equiv s\sigma_p + \mu_f C_0. \tag{19}$$

while the value for θ at the market portfolio is

$$\theta_m = \frac{c\left(\frac{a-b\mu_f}{b-c\mu_f}C_0\right) - bC_0}{d}\Sigma^{-1}\mu + \frac{aC_0 - b\left(\frac{a-b\mu_f}{b-c\mu_f}C_0\right)}{d}\Sigma^{-1}\bar{1}$$

$$= \Sigma^{-1}\left(\mu - \mu_f\bar{1}\right)\frac{C_0}{b - c\mu_f} \tag{20}$$

2.5 OPTIMAL PORTFOLIO

This is the best combination of the risk free asset and the market portfolio..Suppose a portion Q_f will be invested in the risk free asset and a proportion Q_m in the market portfolio. So the portfolio return become:

$$R_p = \Theta_f R_f + \Theta_m R_m$$

and the utility function:

$$E(C_0 + R_p) - \frac{1}{2}¥\ \mathrm{var}(C_0 + R_p) = C_0 + \Theta_f R_f + \Theta_m «_m - \frac{1}{2}¥\Theta_m^2 ±_m^2$$

(21)

So the optimization problem is:

$$Max\left\{ C_0 + \Theta_f R_f + \Theta_m «_m - \frac{1}{2}¥\Theta_m^2 ±_m^2 \;|\; \Theta_f + \Theta_m = 1 \right\}$$

So the vector of the amounts the investor should invest in each individual asset , portfolio mean and standard deviation are respectively :

$$\bar{–}_{opt} = \begin{pmatrix} \bar{–}_1 \\ \vdots \\ \bar{–}_N \\ \bar{=}_f \end{pmatrix} = \begin{pmatrix} \frac{1}{¥}\sum^{-1}\left(« - «_f \bar{1}\right) \\ C_0 - \frac{b - c«f}{¥} \end{pmatrix} \qquad (22)$$

$$«_{opt} = «^T \frac{1}{¥}\left(\sum^{-1} « - «_f \sum^{-1}\bar{1}\right) + «_f\left(C_0 - \frac{b - c«f}{¥}\right)$$

$$= \frac{1}{2}\left(c«_f^2 - 2b«_f + a\right) + «_f C_0 \equiv \frac{1}{2}s^2 + «_f C_0$$

(23)

$$\sigma_{opt} = \sqrt{\left(\frac{1}{\gamma}\left(\Sigma^{-1}\mu - \mu_f\Sigma^{-1}\bar{1}\right)\right)^T \Sigma \left(\frac{1}{\gamma}\left(\Sigma^{-1}\mu - \mu_f\Sigma^{-1}\bar{1}\right)\right) + 0}$$

$$= \frac{1}{\gamma}\sqrt{c\mu_f^2 - 2b\mu_f + a} \equiv \frac{1}{\gamma}s \qquad (24)$$

Example

Let's consider an investor has 1 euro to invest in some(seven) securities[21,1] , C0=1 and suppose we collect returns (daily,. weekly or monthly) for a period (1 year or more)

Basing on these data we calculate the vector of mean returns and the covariance matrix [21] [5](daily, weekly or monthly) with a C++ code.

Table 1. Expected rate of return on asset

m_i -	0.280	0.197	0.125	0.320	0.280	0.195	0.284

Table 2. The covariance matrix

Matrix of covariances of r							
Assets	1	2	3	4	5	6	7
1	0.380	0.202	0.145	0.120	0.125	0.089	0.060
2	0.140	0.205	0.241	0.098	0.224	0.184	0.128
3	0.125	0.182	0.125	0.124	0.317	0.114	0.093
4	0.120	0.121	0.116	0.198	0.121	0.089	0.052
5	0.280	0.102	0.425	0.085	0.347	0.178	0.150
6	0.145	0.189	0.113	0.111	0.314	0.145	0.085

7	0.084	0.161	0.101	0.127	0.100	0.045	0.184

Next are calculated the amounts invested in different portfolios given in the mentioned tables as well as the respective the mean and the covariances for example for portfolio with minimum variances.Tab 3 It is to be mention we use only synthetic data so the details and analysis on the effectiveness of different portfolios has not been carry out.

Table 3 The amounts invested at the portfolio with minimum variance

te1[1]=-0.336	te2[1]=1501
te1[2]=1.235	te2[2]=-4000
te1[3]=0.542	te2[3]=-2096
te1[4]=0.082	te2[4]=1146
te1[5]=0.336	te2[5]=-977.7
te1[6]=-0.99	te2[6]=3940
Q_{EF} =te1+m_P * te2 The mean and minimum variance mymv=0.0002 simv=0.01138	

Table 4. The amounts invested at different portfolios

temv[1]=0.0954	tetg[1]=0.5039	teop1[1]=0.549	tem[1]=0.993	teopt[1]=0.5024
temv[2]=0.0828	tetg[2]=-1.006	teop1[2]=-1.127	tem[2]=-2.31	teopt[2]=-1.168
temv[3]=-0.061	tetg[3]=-0.632	teop1[3]=-0.695	tem[3]=-1.31	teopt[3]=-0.665
temv[4]=0.4120	tetg[4]=0.7238	teop1[4]=0.758	tem[4]=1.097	teopt[4]=0.5549
temv[5]=0.0548	tetg[5]=-0.211	teop1[5]=-0.240	tem[5]=-0.53	teopt[5]=-0.268
temv[6]=0.1442	tetg[6]=1.216	teop1t[6]=1.336	tem[6]=2.501	teopt[6]=1.265
temv[7]=0.2723	tetg[7]=1.009	teop1[7]=1.091	tem[7]=1.891	teopt[7]=0.9564
				teopt[8]=0.4944

where Q_{EF} - allocation when when the portfolio is on the efficient frontier ,temv-allocation at the minimum variance portfolio,tetg - allocation at the tangency portfolio,
teop1-allocation at the Sharpe portfolio,tem - allocation at the market portfolio,teopt2-allocation having the risk free asset.. It is to be mention we use only synthetic data so the details and analysis on the effectiveness of different portfolios has not been really carry out.

3.THE MODEL SOLUTION VIA INTERIOR – POINT METHOD.

Generally, the quadratic programming problems are formulated as minimization
[14,7] Therefore we shall consider problems given in the following form:

$$\min c^T x + \frac{1}{2} x^T Qx$$

$$\text{subject} \quad Ax \geq b \qquad x \geq 0 \tag{25}$$

where the matrix Q is symmetric.

We introduce a nonnegative vector w of surplus variables and then substract a barrier term [4] for each nonnegative variable to get the following barrier problem:

$$\min c^T x + \frac{1}{2} x^T Qx - « \sum_j \log x_j - « \sum_i \log w_i \tag{26}$$

sub $Ax - w = b$

Considering interior – point method.[14] we replace (x, w, y, z) with..Next we drop the nonlinear terms to get the following linear system for the step directions $\Delta x, \Delta w, \Delta y, \Delta z$:

$$\begin{bmatrix} -(X^{-1}Z + Q) & A^T \\ A & Y^{-1}W \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = \begin{bmatrix} c - A^T y - «X^{-1}e + Q_x \\ b - Ax + «Y^{-1}e \end{bmatrix} \tag{27}$$

4.BEYOND THE MARKOWITZ MODEL

So far we have discussed the portfolio optimization, in which the risk is measured by standard deviation. There is some criticism against this approach, because standard deviation is a measurement of a volatility, which is both upside and downside.

In addition there are other models, for example : Roy, Kataoka, and Tesler in which the downside risk is the safety first principle., or other random ones (with elliptical distribution etc) , portfolios that use Value at risk measure or other objective functions as so called Economic value added (EVA) and The Risk Adjusted Return On Capital (Raroc).

Also some inputs of the Markowitz models are random quantities they could be studied as random models etc., modeling uncertainty [18] of input parameters etc basing on the stochastic portfolio theory which is descriptive rather than normative. This theory can model any market , because it is compatible with or without arbitrage, with equilibrium or disequilibrium etc, while the normative approaches (including Markowitz) assume that certain norms or requirements as non- arbitrage, market equilibrium and completeness are satisfied by the market.

5. CONCLUSION.

In our country it is the time "to put into computer " and optimize the most important investment basing on optimization ,specifically on portfolio theory and their models , even more to treat them in a general sense , beyond Markowitz., basing on optimization theory ,Stochastic Portfolio Theory, functional analysis, stochastic ,convex (cone) analysis, numerical analysis methods as interior point (KKT etc) etc..

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