Finiteness Conditions for Clifford Semigroups

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ABSTRACT

There is a large amount of published work in the last decade on finiteness conditions of monoids and groups such as FP_n and its siblings. Recently Gray and Pride have found that a Clifford monoid containing a minimal idempotent e is of type FP_n if and only if its maximal subgroup containing e is of the same type. In our paper we look for results which are in the same spirit as the above, that is, we try to relate the homological finiteness conditions of a Clifford monoid to those of a certain group arising from its semilattice structure. More specifically, we prove that if a commutative Clifford monoid S is of type FP_n, then its maximum group image G is of the same type. To achieve this we employ a result of [10] which relates the cohomology groups of S to those of G, and the fact that the functor $Ext^n_{\mathbb{Z}S}(\mathbb{Z},\bullet)$ commutes with direct limits whenever S is of type FP_n.

INTRODUCTION

Although semigroups appear almost everywhere in everyday algebra, they are much harder to work with compared with structures like groups for example. This is the reason why authors tend to study those semigroups which are close to groups in certain aspects. There is a subclass of semigroups, called Clifford semigroups , which has attracted special interest in the last couple of years for several reasons. First, Clifford semigroups are decomposed into groups in a nice way, second, properties of the groups into which a Clifford semigroup decomposes, give rise to similar properties of the semigroup itself, third, as proved in [10], Clifford semigroups can be identified with certain functors with codomain **Grp** whose colimit group has cohomology groups nicely related with the cohomology groups of the semigroup itself. In this paper we use this relation to relate certain homological properties of a Clifford monoid to those of the colimit group of the functor arising from it. Before we go on further with proving our result, we give the necessary definitions and results which we use to make the proof. First let explain what

Clifford semigroups are. The following theorem gives equivalent definitions of it and is found in [4].

Theorem 1.1 *Let S be a semigroup and E its set of idempotents.Then the following statements are equivalent:*

(1) S is a Clifford semigroup;
(2) S is a semilattice of groups;
(3) S is a strong semilattice of groups;
(4) S is regular, and the idempotents are central;
(5) S is regular, and $\mathcal{D}^S \cap (E \times E) = 1_E$.

If E is a semilattice, then a strong semilattice of groups is a collection of groups $\{H_e \mid e \in E\}$ together with group homomorphisms $\overline{\ }_{e,e'} : H_e \to H_{e'}$ for every $e \geq e'$ satisfying the following conditions.

1. For every $e \in E, \overline{\ }_{e,e'}$ is the identical homomorphism of H_e ,
2. for every $e_1 \geq e_2 \geq e_3, \overline{\ }_{e_2,e_3} \overline{\ }_{e_1,e_2} = \overline{\ }_{e_1,e_3}$.

It turns out that if $e \geq e'$ and $f \in E$ such that $e' = ef$, then $\overline{\ }_{e,e'} : H_e \to H_{e'}$ sends every $x \in H_e$ to xf . In other words we have that

$$\overline{\ }_{e,e'} = \circ_f \mid H_e$$

where $\circ_f$ is the right translation of S by f .

We can express the multiplication in Clifford semigroups S in terms of mappings $\overline{\ }_{e,e'}$ and the multiplication in each group H_e in the following way. For every $e_1, e_2 \in E$ and every $x_1 \in H_{e_1}$ and $x_2 \in H_{e_2}$, we have

$$x_1 x_2 = \overline{\ }_{e_1,e_1e_2}(x_1)\overline{\ }_{e_2,e_1e_2}(x_2) = (x_1e_2)(x_2e_1)$$

where the multiplication of the right hand side is the multiplication of the group $H_{e_1e_2}$.

Lastly, note that groups H_e of the theorem are the Green's $\mathcal{H}$ -classes which separate the idempotents of S .

In [10] every Clifford semigroup S with the semilattice of idempotents E is regarded as a functor with domain E and codomain **Grp**. It turns out that the colimit of that functor is exactly the maximum group image G of the semigroup and if S is commutative, there exists a natural isomorphism

$$Ext^n_{\mathbb{Z}G}(\tilde{\boldsymbol{J}}N, M) \cong Ext^n_{\mathbb{Z}S}(N, \boldsymbol{J}^*M) \qquad (1)$$

for every $M \in \boldsymbol{Ab}^{\mathbb{Z}G}$ and $N \in \boldsymbol{Ab}^{\mathbb{Z}S}$. Here $\tilde{\boldsymbol{J}}$ is a left adjoint to the functor $J^* : \boldsymbol{Ab}^G \to \boldsymbol{Ab}^S$ defined by the rule $J^*(M) = MJ$, for every G-module $M \in \boldsymbol{Ab}^G$.

In [3] the authors prove that a Clifford monoid containing a minimal idempotent e is of type FP_n if and only if its maximal subgroup containing e is of the same type. This is a very nice result and it is what inspired us to look for similar results but which do not really involve homological properties of a particular group from the semilattice structure of the semigroup but of a group which captures information from the whole structure. In fact such a group is the maximum group image of the Clifford semigroup and as it turns out, it has to be of type FP_n whenever S is of the same type. It remains open whether the converse is true or not but we do believe that it also holds true.

If this assumption is right, it amounts to say that the study of certain homological properties of Clifford commutative monoids reduces to the study of the same properties of their maximum group image.

A monoid S is called of type FP_n for some $n \geq 1$ if there is a free finitely generated partial resolution of $\mathbb{Z}S$-modules of the trivial $\mathbb{Z}S$-module $\mathbb{Z}$;

$$C_n \xrightarrow{\ ¢_n\ } C_{n-1} \xrightarrow{\ ¢_{n-1}\ } \ldots \xrightarrow{\ ¢_2\ } C_1 \xrightarrow{\ ¢_1\ } C_0 \xrightarrow{\ ¢_0\ } \mathbb{Z} \longrightarrow 0,$$

or equivalently, if the trivial $\mathbb{Z}S$-module $\mathbb{Z}$ is of type FP_n. It is proved in [1] that for an R-module M being of type FP_n is equivalent to the condition that the functor $Ext^n_R(M, \bullet)$ preserves direct limits for all n.

Main result

Before we give our result we recall from [10] that regarding monoids S and G as small categories with a single object, denoted by $*_S$ and $*_G$ respectively, and morphisms, the elements of the respective monoids, and denoting them additively, one can define $J : S \to G$ by $J(*_S) = *_G$ and for every $s \in S$ we let $J(s) = s + K$. It is not difficult to see that J is a functor (a homomorphism of monoids in this case). The functor J induces a functor $J^* : \boldsymbol{Ab}^G \to \boldsymbol{Ab}^S$ by the rule $J^*(M) = MJ$, for every G-module $M \in \boldsymbol{Ab}^G$. It is proved in theorem 4.1 of [10] that J^* has a left adjoint $\tilde{J}$ defined by the rule $\tilde{J}(T) = Lan_J T$ for every functor $T \in \boldsymbol{Ab}^S$. Here $Lan_J T$ is the left-Kan extension of T along J and is defined by

$$Lan_J T(*_G) = \underrightarrow{co\lim}(\Im \downarrow *_G \xrightarrow{P} S \xrightarrow{T} \boldsymbol{Ab}) \qquad (2)$$

where $\Im \downarrow *_G$ is the comma category of J-objects over $*_G$ and P is the projection $(*_S, a) \mapsto *_S$. In this setting we can prove the following

Theorem 2.1 *Let S be a Clifford commutative monoid and G its maximum group image. If S is of type FP_n, then G is of type FP_n.*

Proof. If in the isomorphism (1) we take N to be the trivial left $\mathbb{Z}S$ module $\mathbb{Z}$, then $\tilde{\boldsymbol{J}}N$ coincides with the trivial $\mathbb{Z}G$ module $\mathbb{Z}$. Indeed, from (2) we see that all the objects $(*_S, a)$ of $\Im \downarrow *_G$ map through the composite $\Im \downarrow *_G \xrightarrow{P} S \xrightarrow{T} \boldsymbol{Ab}$ to $\mathbb{Z}$, and all the morphisms $s : (*_S, a) \to (*_S, b)$ map to $id_{\mathbb{Z}}$, therefore the colimit of the resulting diagram is just $\mathbb{Z}$.

Thus (1) now makes

$$Ext^n_{\mathbb{Z}G}(\mathbb{Z}, M) \cong Ext^n_{\mathbb{Z}S}(\mathbb{Z}, \boldsymbol{J}^* M). \qquad (3)$$

To prove that G is of type FP_n, as shown in [1], we need to prove that $Ext^n_{\mathbb{Z}G}(\mathbb{Z}, \bullet)$ commutes with direct limits. Let $\underrightarrow{\lim} M_i$ be a direct limit of a diagram of modules M_i with $i \in I$.

$$
\begin{aligned}
Ext^n_{\mathbb{Z}G}(\mathbb{Z}, \varprojlim M_i) &\cong Ext^n_{\mathbb{Z}S}(\mathbb{Z}, \boldsymbol{J}^* \varprojlim M_i) && \text{from (3)} \\
&\cong Ext^n_{\mathbb{Z}S}(\mathbb{Z}, \varprojlim \boldsymbol{J}^* M_i) && \text{right adjoints preserve limits} \\
&\cong \varprojlim Ext^n_{\mathbb{Z}S}(\mathbb{Z}, \boldsymbol{J}^* M_i) && S \text{ is of type } FP_n \\
&\cong \varprojlim Ext^n_{\mathbb{Z}G}(\mathbb{Z}, \boldsymbol{J}^* M_i) && \text{from naturality of (3)}
\end{aligned}
$$

therefore G is of type FP_n as required. □

Problem 1 *Does the converse of Theorem 2.1 holds true?*

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