Some Properties of Semigroups

Edmond PISHA[1], Osman HYSA[2]
1)Department of Mathematics, Faculty of Natural Sciences,Universityof Tirana, Tirana, Albania.
Email: pishamondi@yahoo.com – Phone: +355692532476
2)Department of Mathematics, "Aleksandër Moisiu" University, Durrës.
Email: o.hysa@yahoo.com – Phone: +355692040155

ABSTRACT
In his paper, Hall T. R, proved some local propertiesfor semigroups.In this paper we provide alternative proofs for some local properties of semigroups studied by Hall. In addition we introducesome new local properties of semigroups and generalize some of the Hall's results. Aslo we give some examples to illustrate the results.

Keywords: Local subsemigroup, regular elements, idempotent elements,weakly commutative semigroup, Green's relations, Archimedian semigroup, local properties.

INTRODUCTION

Definition 1.1 [4] A Local subsemigroup of a semigroup *S* will be called every subsemigroup of form *eSe*, where *e* is an idempotent element of *S*.

Nambooripad in Decalb's Conferenca 1979 raised the problem: If S is a regular semigroup such that for every idempotent $e \in S$, idempotents of eSe form a band, then idempotents of eS (or Se) form a band too.

The answer of this question was given in affirmative form Hall [3]. Before continuing with presenting the results of Hall we will explain some concepts. We assume that the reader is familiar with Green's relations **L**, **R**, **H**, **D** in the form they are defined by Clifford A. H. and Preston G. B. [1].

Definition 1.2. [1] By Reg(S) we denote the set of all regular elements of semigroup S.

Definition 1.3. By V(x) we denote the set of all inverse elements of x in S.

Definition 1.4. The (p, q, r)-regular element of semigroup S we call an element a of S such that $a \in a^pSa^qSa^r$ (p, q, r are non negative integers).

Definition 1.5. By (p, q, r) – Reg(S) we denote the set of all (p, q, r)-regular elements of semigroup S.

Definition 1.6. The element a of semigroup S will be called right (left) divisible from element b, if and only if $a = xb$, $x \in S$ ($a = by$, $y \in S$). The element a will be called divisible from element b if it is left and right divisible from b.

Definition 1.7. A semigroup S is called right archimedian if for two elements $a, b \in S$ there is a power of a such that is divided by b and there is a power of b that is divided by a.

Definition 1.8. A semigroup S is called weakly commutative if for $a, b \in S$ hold $(ab)^k = xa = by$ for any $x, y \in S$ and k positive integer number.

Definition 1.9. A semigroup S is called bi-simple if it contains only one D-class.

For other concept we use here the reader may refer the book of Clifford and Preston.

We will use these results:

Proposition 1.1. (Lemma 2.14 [1]) Every element e of a semigroup S is right neutral element for Le, left neutral element for Re and neutral element for He.

Proposition 1.2. (Theorem 2.17 [1]) If a and b are elements of semigroup S then $ab \in R_a \cap L_b \Leftrightarrow R_b \cap L_a$ has idempotent element. In this case $aH_b = H_a b = H_a H_b = H_{ab} = R_a \cap L_b$.

Proposition 1.3. (Lemma 2.13 [1]) If a is a regular element of semigroup S, then $aS^1 = aS$ and $S^1 a = Sa$.

Proposition 1.4. (Theorem 2.16 [1]) If a, b and ab are elements of a H-class H of semigroup S, then H is subgroup of S. In particular, every H-class that contain an idempotent element is a subgroup of S.

Finally, we note that if $x' \in V(x)$, then $x \mathrm{L} x'x$ and $x \mathrm{R}\, xx'$.

Now we give some alternative proofs of T. E. Hall results.

Theorem 1.5. (*i*) If Reg(eSe) is a subsemigroup generated by idempotents, then Ref(eS) is also a subsemigroup generated by idempotents.

(*ii*) If Reg(*eSe*) is subsemigroup in which H is congruence, then Reg(*eS*) is subsemigroup in which H is congruence.

(*iii*) If Reg(*eSe*) is union of groups (non necessarily a subsemigroup), then and Reg(*eS*) is union of groups.

(*iv*) If *eSe* has at most one idempotent for every L-class, then *eS* has at most one idempotent for every$L$-class.

Proof.

(*i*) If Reg(*eSe*) is subsemigroup, then, as it is showed by Hall [3], Reg(*eS*) is subsemigroup also. Now, it is clear Reg(*eS*) $\subseteq$ Reg(*eSe*)E(*eS*). Since g(*eSe*) is generated by idempotents, then Reg(*eS*) is generated by idempotents also.

(*ii*) Since Reg(*eSe*) is subsemigroup, then, as is showed by Hall [3], Reg(*eS*) is subsemigroup also. Let us prove that H is congruence in Reg(*eS*). Let p and q be elements from Reg(*eS*) such that p H q. Let us show that $\forall c, d\in$ Reg(*eS*), pcHqc and dpH dq. From pH$q \Rightarrow p$L$q \wedge p$Rq. Since L is right congruence pcLqc, so peLqe. It is easy to show that from pR q we have pe R p R qR qe (since it is easy to show pe R p and q R qe). Now it is clear that pe R qe, which combined with pe L qe implies pe H qe. From conditions we have pce H qce (for $ce\in$ Reg(*eSe*)). Thus pce L qce, pce R qce. So pce R pc, qce R qc therefore pc R qc.

From pc L qc and pc R qc we have pc H qc.

Similarly p H q imply p L q and p R q and since Ris left congruencewe will have pd R dq. Let’s show now pd L dq. We have Reg(*eSe*) = Reg(*eS*)*e*so p L q imply Reg(*eS*)p = Reg(*eS*)$q$and then Reg(*eSe*)pe = Reg(*eSe*)qe. Now Reg(*eSe*)$depe$ = Reg(*eSe*)$deqe$ thus Reg(*eS*)dpe = Reg(*eS*)dqe; So Reg(*eS*)dp = Reg(*eS*)dq (because $q\in$ Reg(*eS*)p). Finally,dp L dq in Reg(*eS*).

From dp L dq and dp R dq results dp H dq. Finally we have shown that H is a congruence in Reg(*eS*).

(*iii*) Let us prove that Reg(*eS*) is union of groups. Since union of subgroups of Reg(*eS*) is subset of Reg(*eS*), then it is sufficient to show the inclusion of Reg(*eS*) in a union of subgroups of Reg(*eS*).

For $a\in$ Reg(*eS*) we prove that $H_a = R_{aa'}\cap L_{a'a}$ has an idempotent. From $ae \in$ Reg(*eSe*) and conditions we have $H_{ae} = R_{aa'e}\cap L_{a'ae}$ has an idempotent. From **Proposition 1.2**, $a'a^2a'e\in R_{aa'e}\cap L_{a'ae}$. Now we have $a'a^2a'$ R $a'a^2a'e$ R $a'ae$ R $a'a$. From $a'a^2a'$ L $aa'e$ and $a'a^2a'$ L aa'. So $a'a^2a'\in R_{aa'}\cap L_{a'a}$. From

Proposition 1.2 $H_a = R_{aa'} \cap L_{a'a}$ has an idempotent and from **Proposition 1.4** we have that H_a is subgruop; so we showed that a is an element of the subgroup H_a.

(*iv*) Let $f, g \in E(eS)$ such that $f\ L\ g$. We must show $f = g$. From $f\ L\ g$it follwos that $fe\ L\ ge$. Since $fe, ge \in E(eSe)$ we have that $fe = ge$. Now we have $f = f\,f = fef = gef = gf$. From $f\ L\ g$ and **Proposition 1.1** we get $gf = g$ and consequently $f = g$.

Now **Theorem 1** (*iv*) of [3] states that if Reg(*eSe*) is semigroup, then Reg(*eS*) is subsemigroup or equivalently: If <E(*eSe*)> is regular subsemigroup, then and <E(*eS*)> is a regular semigroup.

In an attempt to generalize this result concerning to (p, q, r)-regularity we introduce this problem:

Problem: If (p, q, r) – Reg(eSe) is subsemigroup, then is (p, q, r) – Reg(eS) subsemigroup?

Answer to this question is partially positive.

We distinct these cases concerning to triples (p, q, r):

First case: (1) $(p, q, r) = (0, 0, 0)$.

Second case: $p \geq 1$; $q, r \geq 0$

(2) (1, 0, 0)	(3) (1, 0, 1)	(4) (1, 1, 0)
(5) $(1, 0, r)$, $r > 1$	(6) (1, 1, 1)	(7) $(1, 1, r)$ $r > 1$
(8) $(1, q, 0)$, $q > 1$	(9) $(1, q, 1)$, $q > 1$	(10) $(1, q, r)$, $q, r > 1$
(11) $(p, 0, 0)$, $p > 1$	(12) $(p, 0, 1)$, $p > 1$	(13) $(p, 0, r)$, $p, r > 1$
(14) $(p, 1, 0)$, $p > 1$	(15) $(p, 1, r)$, $p, r > 1$	(16) $(p, q, 0)$, $p, q > 1$
(17) $(p, 1, 1)$, $p > 1$	(12) $(p, q, 1)$, $p, q > 1$	(19) (p, q, r), $p, q, r > 1$

Third case: $p = 0$; $q, r \geq 0$

(20) (0, 0, 1)	(21) $(0, 0, r)$, $r > 1$	(22) (0, 1, 0)
(23) $(0, q, 0)$, $q > 1$	(24) (0, 1, 1)	(25) $(0, 1, r)$, $r > 1$
(26) $(0, q, 1)$, $q > 1$	(27) $(0, q, r)$, $q, r > 1$.	

Theorem 1.6. If (0, 0, 0) – Reg(*eSe*) is subsemigroup, then so is and (0, 0, 0) – Reg(*eS*).

Proof. Let suppose a, $b \in$ (0, 0, 0) – Reg(eS), then $a = x_1x_2$ for $x_1, x_2 \in eS$ and $b = y_1y_2$ for $y_1, y_2 \in eS$. Now $ab = (x_1x_2)\ (y_1y_2) = (ab)°(x_1x_2)\ (ab)°(y_1y_2)\ (ab)° \in (ab)°(eS)\ (ab)°(eS)\ (ab)°$.

So $ab \in$ (0, 0, 0) – Reg(eS).

Thus for the first case the response of the question is positive.

The response for second case will give this

Theorem 1.7. If (p, q, r) – Reg(*eSe*) is subsemigrup, then so is and (p, q, r) – Reg(*eS*) for p ñ 1 and q, r ñ 0.

Proof. Let a, $b \in$ (p, q, r) – Reg(eS). If $p \geq 1$, then $b \in b(eS)$ and $b = bec$ for any
$c \in S$. We will have ae, $be \in$ (p, q, r) – Reg(eSe). So $abe = (ae)(be) \in$ (p, q, r)– Reg(eSe).

Thus $abe \in Te$, where $T = (ab)^p\,(eS)\ (ab)^q\,(eS)\ (ab)^r$ and therefore $ab = abec \in Tec \subseteq T$.

Also the response of the question is positive for the second case.

For the third case we will give an example, which shows that the answer is negative for this case.

Example 1.8. Let F be a free monoid over the alphabet {0, 1 and δ the empty word. For $x, y \in F$will have $x\rho y \Leftrightarrow u\ 010\ vy = u\ 1v$ for $u, v \in F$ and $x\sigma y$, if and only if there exists a string $x = x_1, x_2, ..., x_n = y$ ($n \geq 1$),where $x_i\rho x_{i+1}$ or$x_{i+1}\rho x_i$ for $i = 1, 2, ..., n - 1$ ($n \geq 2$). It is clear σ is congruence in F.

Let us denote by $T = F/\sigma$ the factor monoid with unity $\Delta = \{\delta\}$. For $X, Y \in T$ we have $XY = \Delta \Rightarrow X = \Delta = Y$. Let us supose that $B, C \in T$, where $1 \in B$ and $0 \in C$. It is evidente that $B = CBC$and $B^2 = XB^2Y \Rightarrow X = \Delta = Y$.

Let us denote by E the zero right semigroup which contain two elements α,β and consider the semigroup ExT (where the operation is coordinatewise). ExT satisfy the condition:

$$(u, X)(u, Y) = (\alpha, \Delta) \Rightarrow (u, Y) = (\alpha, \Delta).$$

Let us denote by $S = (EXT) \setminus \{(\alpha, \Delta)\}$ and $e = (\beta, \Delta)$, $b = (\alpha, B)$, $c = (\alpha, C)$. We then have $e^2 = e$, $eb = b$ and $ec = c$. It is evident that the semigroup S is generated by e, b, c and $eS = S$, $eSe = Se$. Now, for the element $x \in Se$ we have $x = exe \in SexSe$ and so

$(0, 1, 0)$ – Reg(eSe) = Se is a semigroup of S. It is clear that $b = cbc$ and so $b \in$ $(0, 1, 0)$ – Reg(eS). Let supposse that $b^2 \in (0, 1, 0)$ – Reg(eS). Then $b^2 \in Sb^2S$ and $b^2 = xb^2y$ for any $x, y \in S$. Thus we have $(\alpha, B^2) = (u, X)(\alpha, B^2)(v, Y)$, where $x = (u, X)$ and $y = (v, Y)$. So $v = \alpha$ and $B^2 = XB^2Y$ hence $Y = \Delta$ and $y = (\alpha, \Delta) \notin S$.

Consequently $b^2 \notin Sb^2S$ and $b^2 \notin (0, 1, 0)$ – Reg(eS). As a result $(0, 1, 0)$ – Reg(eS) is not a subsemigroup of S.

The following theorems give some other local properties of semigroups.

Theorem 1.8.

(*i*) If *eSe* is a bisimple subsemigroup, then *eS* is also a bisimple subsemigroup.

(*ii*) If Reg(*eSe*) is a zero right semigroup, then Reg(*eS*) is also a zero right semigroup.

Proof. (*i*) Let D_a and D_b be two D-classes of eS, where $a, b \in eS$. We must prove $D_a = D_b$.

Let $p \in D_a$, then $p\ D\ a$ or $p\ L \circ R a$ and $p\ L\ z$, $z\ R\ a$ for any $z \in eS$. We will have $pe\ L\ ze$ and $ze\ R\ ae$ (since $z\ R\ ze$ and $a\ R\ ae$); finally we have $pe\ D\ ae$. From the assumption we have that $D_{ae} = D_{be}$, so $pe\ D\ be$ which shows that $pe\ L\ k$ and $k\ R\ be$ for any $k \in eSe$. It is clear that $be\ R\ b$. From $pe\ L\ k$ and $k\ R\ b$ it follows that $pe\ D\ b$. Further, from $pe\ R\ p$ we have that $pe\ D\ p$ and then $p\ D\ pe\ D\ b$, $p\ D\ b$, $b \in D_b$. So, we have showed that $D_a \subseteq D_b$. Similarly we prove $D_b \subseteq D_a$ and then $D_a = D_b$.

(*ii*) Since Reg(eSe) is subsemigroup, then, as it is shown by Hall [3] **Theorem 1**-(*iv*), Reg(eS) is a subsemigroup. If $p, q \in$ Reg(eS), then we have

$$pq = pqq'q = pqeq'q = qeq'q \text{ (from the assumption) } = qq'q = q.$$

Thus, $pq = q$ for all $p, q \in$ Reg(eS), which means that Reg(eS) is zero right semigroup.

Theorem 1.9. In weakly commutative semigroup *S*, if *eSe* is a right (left) archimedian subsemigroup, then *eS* is aright (left) archimedian subsemigroup.

Proof. Let *eSe* be a right archimedian subsemigroup of semigroup *S* and let us show that *eS* is also a right archimedian subsemigroup of *S*. If a, $b \in S$, then ae, $be \in eSe$. Since *eSe* is right archimedian, then $(ae)^n = x(be)$ and $(be)^n = y(ae)$ for any n, m positive integers and

$x, y \in eSe$. Since $a^{n+1} = a^n a = (ae)^a a$ (as $a \in eS$) $= x(be)a = xba$, so $a^{n+1} = xba$.

Thus, $a^{(n+1)p} = (xba)^p = u(xb)$ for any $u \in eSe$ and p – positive integer. Similarly, $b^{m+1} = b^m b = (be)^m b = (yae)b = yab$ and $b^{(m+1)q} = (yab)^q = v(ya)$ for any $v \in eSe$ and q – positive integer.

Thus we have proved that *eS*is right archimedian. In a similar fashion we prove the case for left archimedian semigroups.

REFERENCES

[1] Clifford AH and Preston GB, (19??) The algebraic theory of semigroups, Math. Surveys, No. 7, Amer. Math. Soc, Providence, R1 Vol. 1.

[2] Hall TE(1973), On regular semigroups, J. Algebra, 24, 1 – 24.

[3] Hall TE (1982) Some properties of local subsemigroups inherited by larger subsemigroups, Semigroup Forum, Vol. 25, 35 – 49.

[4] Mc Alister DB, (1981) Regular Rees matrix semigroups and regular Dubreil – Jacotin semigroups, J. Austral. Math. Soc. (Series A), 31, 325 – 336.