The Fundamental Theorems Of Calculus For The M_β −Integral

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Abstract

One version of Mcshane integral with respect to a basis for the function with value in Dedekind complete Riesz space is introduced. The fundamental theorems of calculus for the M_β −integral are proved

Keywords:Mcshane and Henstock integration ,Riesz spaces,Derivation basis,Fundamental theorem of calculus

Introduction and preliminaries

In this paper we introduce the Mcshane integration of Riesz-valued functions in the case of the derivation basis.We demonstrate also the classic fundamental theorems of Calculus.The paper is structured as follow.In Section 2 we recall some fundamental concepts related to Riesz spaces and derivation bases.In Section 3 we investigate some basic properties of Mcshane integral with respect to a Basis.In Section 4 we demonstrate the classic Fundamental theorems of Calculus.

Definition2.1*We say that a net $(r_\beta)_\beta$ order converges (or short $(o)-$converges) to $r \epsilon R$ if there exists an $(o)-net$ $(p_\beta)_{\beta \epsilon A}$ satisfying $|r_\beta - r| \leq p_\beta$ for each $\beta \epsilon A$ we shall write in this case $r = (o)lim_\beta r_\beta$.In case A= N we get definition of (o) convergent sequence.*

Definition2.2 *We say that a net $(r_\beta)_{\beta \epsilon A}$ is $(r)-converges$ for $r \epsilon R$ if there exists $\beta_0 \epsilon A$ so that $|r_\beta - r| \leq \epsilon u$ for all $\beta \geq \beta_0$.In case A= N we get the definition of (r) convergent sequence.*

Definition2.3 *A Dedekind complete Riesz space is said to be regular if it satisfies property σ and if for each sequence $(r_n)_n$ in R,order convergent to zero,there exists a sequence $(k_n)_n$ of positive numbers ,with $lim_n k_n = +\infty$,such that the sequence $(k_n r_n)_n$ is order convergent to zero.*

***Proposition* 2.4**[4](Theorem1, p. 350) *In a regular Riesz space the $(o)-$convergence is equivalent to the$(r)-$convergence .*
An interval is always a compact nondegerate subinterval of R.if $E \subset R$,then $|E|$denotes the Lebesgue measure of E.
A collection of intervals is called non overlapping if their interiors are pair wise disjointed. *Let be* $[a, b]$a fixed interval of R and $\mathcal{Z}$ the family of all subintervals of $[a, b]$. An $M-$basis on $[a, b]$
Is by definition, any subset $B(M)$ *of* $\mathcal{Z} \times [a, b]$such that $(I, x)\epsilon B(M)$implies $x\epsilon E$ for $E \square [a, b]$,
and is denoted with $B_m[E]$.Given a base $B(M)$,an interval I is called a $B(M)-$ interval if $(I, x)\epsilon B(M)$, for any $x\epsilon E$ .We assume that $[a, b]$is a $B(M)-$ interval.For $E \neq \square$and $E \square [a, b]$we denote by Δ_E the directed set of all posive real –valued functions defined on E and endowed with natural ordering:given two functions $\delta_1$and δ_2 from $\Delta_E$we say $\delta_1 \leq \delta_2$ if and only if $\delta_1(x) \leq \delta_2(x)$ for every. $x\epsilon E$
A funksion $\delta: E \to]-\infty, +\infty[$of $\Delta_E$is rreferred as a gauge on E.
For a given gauge $\delta\epsilon\Delta$ we denote

$$B[E] = \{(I, x)\epsilon B(M): I \square]x - \delta(x), x + \delta(x)[, x\epsilon E\}$$

We note that B_δ is also basis on $[a, b]$.
We say that a basis $B(M)$ is a Vitali basis if for any $\delta\epsilon\Delta$ and $x\epsilon[a, b]$the set $B_\delta[x]$ is non empty.
Let be $E \neq \emptyset$ and $E \subset [a, b]$.A
finite subset P of $B_m[E]$ is called $B(M)-$ decomposition on E if for every distinct elements$(I'x')$ and (I'', x'') of P the corresponding intervalsI' and $I''$are non overlapping and $E = \bigcup_{(I,x)\epsilon P} I$.If $E = [a, b]$we say that P is $B(M)-$ partition of $[a, b]$.
Given a gauge δ a $B_\delta(M)-$ decomposition is called $\delta-$ fine . In this paper we assume that each basis $B(M)$considered here is a Vitali basis and has two partitions properties:(a)For any $B(M)$ – interval I and a gauge δ on I there exist a $B_\delta(M)-$ partion of I.If $I_1$and $I_2$are $B(M)$ – intervals and $I_1 \subset I_2$then $I_1 \backslash I_2 = \bigcup_{i=3}^{n} I_i$where $I_i$are non overlapping $B(M)$ – intervals.

If $f:[a,b] \to R$ and P= $\{(J_i, \xi_i): i = 1,2, \ldots, m\}$is a partitions of $[a,b]$the sum $\sum_{i=1}^{n} f(\xi_i) |J_i|$will denoted by $S(f,P)$

3 The Mcshane integral with respect to a Basis

WE now introduce a Mcshane integral type with respect to a basis for Riesz-space valued functions.

Definition 3.1*Let $B(M)$be a fixed basis on$[a,b]$ .We say that $f: [a,b] \to R$ is Mcshane integrable on a $B(M)$ – interval $(B(H) - interval)$ $E \subset [a,b]$ wth respect to $B(M)$ (brief M_B – integrable) if there exist an element $Y \epsilon R$ such that* $\inf_{\delta \epsilon \Delta}(sup\{|\sum_{(I,x)\epsilon P} f(x)\,|I| - Y|: P is\ a\ B_\delta(M) - Partition\ of\ E\}) = 0$ (1)*In this case we write* $(M_B)\int_E f = Y$.

Proposition3.1 *Let R be a Dedekind complete Riesz space ,satisfying property σ,$Q \subset [a,b]$ be countable set, and $f:[a,b] \to R$ be a function, such that $f(x) = 0$ for all $x \epsilon [a,b]\backslash Q$.Then $(M_B)\int_a^b f = 0$*The prof of this preposition can adopted without differences the proof made by [2]for the Henstock integral.

Proposition 3.2*Let R be a Dedekind complete Riesz space and solid ,satisfying property σ,$Q \subset [a,b]$ be a set with $|Q| = 0$, and $f:[a,b] \to R$ be a function, such that $f(x) = 0$ for all $x \epsilon [a,b]\backslash Q$.Then $(M_B)\int_a^b f = 0$*

Proposition3.3 *Under the above condition,the function f is M_B –integrrable on a $B(M)$ –interval E if and only if*

$$\inf_{\delta \epsilon \Delta}[sup\{|S(f,P_1) - B(f,P_2)|: P_1\ and\ P_2 are\ B(M) - partition\ oof\ E\}] =$$

Taking in account of respective property for the Henstock-integral on Riesz space (*see*[2]),we can prove the proposition.

Proposition 3.4 *If $[a,b]$, $[a,c]$ and $[c,b]$ are $B(M)$ –intervals and f is M_B-integrable on $[a,c]$ and on $[c,b]$,then f is also M_B –integrable on $[a,b]$ and*

$$(M_B)\int_a^b f = (M_B)\int_a^c f + (M_B)\int_c^b f$$

Proposition 3.5 *If function f is M_B – integrable on a B –interval of $[a,b]$,then f is also M_B –integrable on any $B(M)$ –intervals I □ $[a,b]$*

Definition 3.2 *A function $f: [a,b] \to R$ has the property $(o) - S^*M(B)$on a $B(M)$ –interval of $[a,b]$ .if*

$$\inf_{\delta\epsilon\Delta_{[a,b]}}\left[\sup_{P}\left\{\sum_{i=1}^{k}\sum_{j=1}^{m}|f(t_i)-f(s_j)|\,|I_i\cap J_j|: P_1,P_2\ are\ B_{\delta(M)}-partit.\ of\ I\right\}\right]$$

Where $P_1=\{(I_it_i), i=1,\dots.k\}$ *and* $P_2=\{(J_js_j), j=1,\dots.m\}$.

Lemma 3.1[4] Let $D=\{(I_i,t_i), i=1,\dots.m\}$ *and* $\mathcal{J}=\{J_j, j=1,\dots.m\}$ *be* $B_{\delta(M)}-partition\ of\ I$

Then $D'=\{(I_i\cap J_j,t_i): i=1,\dots.k; j=1,\dots.m; I_i^0\cap J_j^0\neq\emptyset\}$

Is a $B_\delta-partition\ of\ I$ *and* $M(f,D)=S(f,D')$

Corollary 3.1 *If* f *is* M_B *–integrable on* $[a,b]$ *and* F *is its indefinite integral then*

$$\inf_{\delta\epsilon\Delta}\left\{\sup_{(I,x)\epsilon P}|f(x)|I|-F(I)|: is\ a\ B_{\delta(M)}-partition\ of\ [a,b]\right\}=0$$

Proposition 3.6 *Let R be Dedekind complete Riesz space.A function* $f:[a,b]\to R$ *is* R *is a* M_B *–integrable on a* $B(M)-interval\ I of\ [a,b]$ *, if and only if, for the* $B_{\delta(M)}$ – partition D $D=\{(I_i,t_i), i=1,\dots.m\}$ *and* $\mathcal{E}=\{(J_j,t_j), j=1,\dots.m\}$ *holds*

$$\inf_{\delta\epsilon\Delta_E}\left\{\sup\left[\sum_{i=1}^{m}\sum_{j=1}^{n}|f(t_i)-f(s_i)|\,|I_i\cap J_j|\right]\right\}=0$$

Proposition3.7 *Let R be a Dedekind complete regular Riesz Space and* $f\geq 0$ *be* $M_B-integrable$ *on a* $B(M)-interval$ *and solid* $E\subset[a,b]$. *Then there exist the function* g *and* h M_B *– intergrable such that* $0\leq g\leq f\leq h$ *and there exist a directed net* $(p_\delta)_{\delta\epsilon\Delta}$ *such that* $(M_B)\int_E|f-g|\leq p_\delta$

Proof. Contstruct the simple funksion

$$f(x)=\begin{cases}\frac{k-1}{2^n} & if\ \frac{k-1}{2^n}u\leq f(x)\leq\frac{k}{2^k}\\ 0 & if\ contrary\end{cases}$$

Where $k=1,2,\dots.,n2^n$ and u unit element of R.. We get that $|f(x)-f_{n(x)}|\leq\frac{1}{2^n}u$ (2)

We have that sequence f_n is (r) – convergent to $f(x)$. If we write in a form $f_n(x)=\sum_{i=1}^{n}k_i\,\chi_{E_i}$. We get that there exists $B(M)$-intervals S_n and positive elements $c_n\epsilon R$ such that $f(x)=\sum_{n=1}^{\infty}c_n\,\chi_{S_n}$ for every xϵE. Moreover f is integrable and $\sum_{n=1}^{\infty}c_n|S_n|=\int_E f<+\infty$ (3)

Since S_n are Lebegue measurable there exist compact sets K_n and open sets G_n of $[a,b]\cap E$ *that* $K_n\subset S_n\subset G_n$. And there exists a $\varepsilon>0$ such that $|G_n\setminus K_n|<\epsilon$

we take $c_n|G_n\setminus K_n|<\frac{u\epsilon}{2^{n+2}}$ From the convergence of the series (2) we get $\sum_{N+1}^{\infty}c_n|S_n|<\frac{\varepsilon U}{4}$

Define $g=\sum_{i=1}^{N}c_n\chi_{k_n}$ and $h=\sum_{i=1}^{N}c_n\chi_{G_n}$.

It is easy to note that $g\le f\le h$

and

$h-g=\sum_{i=1}^{N}c_n\chi_{G_n\setminus K_n}+\sum_{i=N+1}^{\infty}c_n\chi_{G_n}\le\sum_{i=1}^{\infty}c_n\chi_{G_n\setminus K_n}+\sum_{i=N+1}^{\infty}c_n\chi_{S_n}$

4 The fundamental theorems of Calculus for the M_B-integral

A function φ is said to be $(o)-$continuous at a point $x_0\in[a,b]$ with respect to basis $-B(M)$ if $\inf_\delta[sup\{|\varphi(I)|:(I,x_0)\epsilon B[\{x_0\}]\}]=0$. Given $E\neq\emptyset$ *and* $E\subset[a,b]$ we say that the function φ is (o)-continuous on E if it is (o)-continuous at every point of E. We say that φ is $(u)-$differentiable on E with respect to basis B if there exists a function g:E→ R such that

$$\inf_{\partial\epsilon\Delta_E}\left[sup\left\{\left|\frac{\varphi(I)}{|I|}-g(x)\right|:(I,x)\epsilon B[E]\right\}\right].$$

The function g is called the $(u)-$derivative with respect to $B(M)$. It is easy to prove that $(u)-$derivative is determined uniquely.

Theorem4. 1 *Let R be a Dedekind complete Riezs space, $B(M)$ a basis and φ be a $R-$valued function on $B(M)-interval$. If φ is $(u)-$ differentiable with respect to $B(M)$ on $[a,b]$ with derivative φ', then φ' is M_B-integrable on $[a,b]$, and*

$$\int_a^b\varphi'=\varphi([a,b]).$$

Proof. By (u) − differentiability of φ in $[a,b]$, there exists an $(o)-$net $(p_\partial)_{\partial\epsilon\Delta}$, Such that $sup\left\{\left|\frac{\varphi(I)}{|I|}-\varphi'(x)\right|:(I,x)\epsilon B_\delta([a,b])\right\}\le(p_\partial),\forall\partial\epsilon\Delta$. Choose a $\partial-$fine partition P= $\{(I_i x_i):i=1,\dots,q\}$ *of* $[a,b],\forall\partial\epsilon\Delta$. From the above inequality, we get $0\le|S(\varphi' P)-\varphi[a,b]|=\left|\sum_{i=1}^{q}\varphi'(x_i)|I_i|-\varphi(I_i)\right|\le\sum_{i=1}^{q}\left\{|I_i|\left|\frac{\varphi(I_i)}{I_i}-\varphi'(x_i)\right|\right\}\le(\sum_{i=1}^{q}|I_i|)p_\delta=(b-a)p_\delta$

Let us follow the idea of [5] for the function Mcshane integrable with real value to prove the theorem:

Theorem4 2 *Let R be a regular Riesz space, B a fixed basis, $f:[a,b]\to R$ and let φ be a $R-$valued function on $B-$interval, such that for some set $Q\subset[a,b]$ with $|Q|=0$. If the* *function f is* $(u)-$ *derivative of* φ *on* $[a,b]\setminus Q$ *with respect to B then f is Mcshane integrable in* $[a,b]$ *and*

$$(M_B) = \int_a^b f = \varphi([a,b]).$$

Proof. Since the Mcshane integrability by virtue of Proposition 3.2 does not depend on values of f on a set of f measure zero,we assume $f(x)=0$ on Q.Let be $Q' = [a,b]\setminus Q$.We can use the Proposition 2.1.As f is the $(u)-$ derivative of φ in Q',then there exist an element $u\geq 0$ of $R, u\neq 0$ such that for every $\varepsilon > 0$ a gauge $\partial_1 \epsilon \Delta_{Q'}$ can be found. If $P = \{(I_i, x_i): i = 1,2,\ldots,q\}$ is a $B_{\partial_1}[Q']$ decomposition, then $\left| |I_i| f(x_i) - \varphi(|I_i|)\right| \leq |I_i|\epsilon u$ For all $i = 1,\ldots,q$. Choose a net (p_β) such $\sum_{j=1}^{n} |I_i| < \varepsilon$,where $I_1, I_2, \ldots, I_n$ are non overlapping intervals with $\sum_{j=1}^{n} \varphi(I_j) < \frac{1}{n} p_\beta$ .Moreover by $(o)-$ continuity of φ with respect to the basis in Q there exists a net (p_β) that

$\sup\{|\varphi([u,v]): x-\delta \leq u \leq x \leq v \leq \delta|\} \leq p_\beta$

We observe $\sum_{(I,x)\epsilon P, x\epsilon E} \varphi(I) \to 0$ .There is gauge δ_2 such that $\sum_{(I,x)\epsilon P, x\epsilon E} \varphi(I) \leq p_\beta$

Put $\delta(x) = min\{\delta_1(x), \delta_2(x)\}$.Then for every B_δ partition $P = \{(I_i, x_i): i = 1,2,\ldots,q\}$of $[a,b]$,$\delta \epsilon \Delta_{[a,b]}$,we have $0 \leq \left|\left[\sum_{i=1}^{q} |I_i| f(x_i)\right] - \varphi([a,b])\right| = \left|\sum_{i=1}^{q} \{|I_i| f(x_i) - \varphi(I_i)\}\right| \leq \left|\sum_{x_i \notin Q} \{|I_i| f(x_i) - \varphi(I_i)\}\right| + \sum_{x\epsilon Q} |\varphi(I_i)| \leq \varepsilon u(b-a) + p_\beta$

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