Some results on Γ-near-field

Eduard Domi
Department of Mathematics
"A.Xhuvani" University, Elbasan
Albania
eduartdomi@hotmail.com

Islam Braja
Department of Mathematics
"A.Xhuvani" University, Elbasan
Albania
braja_islam@yahoo.com

***Abstract*— In this paper we give some results on gamma-near-fields through a new definition. We prove that a Γ-near-field is B-simple and for every α exists an element that is α-distributive and for every 0 ≠m∈M exists an m'∈M such that m'α m ≠0.**

Keywords—Gamma-near-rings, gamma – near – fields

I. INTRODUCTION

Let consider M and Γ as two non empty sets. Every map of M x Γ x M in M is called Γ- multiplication in M and is denoted as $(\cdot)_\Gamma$. The result of this multiplication for elements a, b ∈ M and $\gamma \in \Gamma$ is denoted a γ b.

According to Satyanarayana [5] one Γ- near-ring is a classified ordinary triple (M, +, $(\cdot)_\Gamma$) where M and Γ are non empty sets, + is a sum in M, while $(\cdot)_\Gamma$ is Γ - multiplication on M such that satisfies the following conditions:

1) (M, +) is a group
2) $\forall$ (a, b, c, α, β) ∈ $M^3 x\Gamma^2$, (aαb)βc = aα(bβc)
3) $\forall$ (a, b, c, α,) ∈ $M^3 x\Gamma$, (a + b)αc = aαc + bαc

Example 1.1 [5]. Let (G, +) be a group, X a non empty set and M a set of all the mapping of X in G. The ordered pair (M, +), where + is a sum of mappings of X in G defined by the equality

(f + g)(x) = f(x) + g(x)

is a non-abelian group when G is non-abelian. Let Γ be a set of all the mappings of G and X. If the product of f γ g is defined by the general composure of f o γ o g for every f, g ∈ M and every $\gamma \in \Gamma$, then it is defined in M a Γ - multiplication, $(\cdot)_\Gamma$ such as for every three elements f_1, f_2, f_3 of M and every two elements α, β of Γ the equalities are true:

$f_1\alpha(f_2\beta f_3) = (f_1\alpha f_2)\beta\, f_3$,

$(f_1 + f_2)\alpha f_3 = f_1\alpha f_3 + f_2\alpha f_3$.

Consequently, (M, +, $(\cdot)_\Gamma$) is one Γ - near-ring.

Example 1.2. If in **example 1** the set X is the retainer of G' of group (G',+), M is the set of all the mappings of G in G' such as f(0) = 0 and Γ is the set of all the mappings of G'in G, again M is a Γ-near-ring in relation to the sum of mappings element per element and Γ-multiplication is defined by the general composition fo γ og for every two elements f, g of M and every element $\gamma \in \Gamma$.

II. PRELIMINARY CONCEPTS AND PROPOSITIONS

Here we will give concepts and we will present same auxiliary propositions, which we will use further in the presentation of the main results of the proceeding.

Let (M, +, $(\cdot)_\Gamma$) be a Γ-near-ring and A, B two subsets of M. We define the set

AΓB = {a γ b ∈ M / a, b ∈ M and $\gamma \in \Gamma$ }.

For simplicity we write aΓB instead of {a}ΓB and similarly AΓb instead of AΓ{b}.

Also for every $\gamma \in \Gamma$ we define

A γ B = {a γ b ∈ M / a, b ∈ M}

and for simplicity we write a γ B and A γ b respectively instead of {a} γ B and A γ {b}.

In [10] is define the set as well as

AΓ* B = {a γ (a' + b) - a γ a' / a, a' ∈ A, $\gamma \in \Gamma$, b ∈ B}

Definition 2.1. A Γ-near-ring M is called zero – symmetric if for every a ∈ M and for every $\gamma \in \Gamma$ we have a γ b = 0.

Γ -near-ring of **example 2** is Γ - near-ring zero – symmetric, whereas the one of **example 1** in general is not zero-symmetric.

Definition 2.2 [2]. Let (M, +, $(\cdot)_\Gamma$) be a Γ-near-ring. A subgroup B of group (M, +) is called bi-ideal of M if

$$B\Gamma M\Gamma M \cap (M\Gamma M)\Gamma * B \subseteq B.$$

Definition 2.3. **A Γ-near-ring is called B-simple if there are no bi-ideal different from zero and from M.**

Proposition 2.4 [2] Let (M, +, $(\cdot)_\Gamma$) be a Γ -near-ring zero–symmetric. A subgroup B of group (M, +) is bi-ideal of M in that case and only then BΓMΓB ⊆ B

An **e** element of Γ-near-ring M is called **identity element** if for every a ∈ M and every $\gamma \in \Gamma$ we have a γ e = e γ a = a.

It is very clear that when Γ -near-ring M has an identity element he is unique.

A Γ-near-ring M is called **Γ-near-field** if it has an identity element, has at least one element different from zero and every element different from zero has a unique inverse element, meaning for every 0 ≠ a ∈ M exists a unique element of a' ∈ M such that a γ a' = a' γ a = e for every $\gamma \in$ Γ, where **e** is an identity element of M.

In the same way, a **d** element of Γ-near-ring (M, +, $(\cdot)_\Gamma$) will be called **α - distributive** if for every two elements a, b of M we have

$$d\alpha (a + b) = d\alpha a + d\alpha b.$$

M_d is a set of distributive elements of M, meaning that the set of elements a∈M such that for every b, c∈M and every

$\gamma \in \Gamma$ we have a γ (b + c) = a γ b + a γ c.

III. Some Results on Gamma-Near-Fields

We will define Γ-near- field similar to the definition of Γ-group given for the first time in [1].

Let (M, +, $(\cdot)_\Gamma$) be a Γ-near –ring and α is a fixed element of Γ. We define in M the operation (α) through the equivalence a(α)b = aαb. It is clear that the operation (α) is commutative and distributive from the right in relation with the sum + in M. Hence, we derive the near-ring (M +(α)) that we denote it simply M_α.

According to Sen and Saha [6] Γ-semi-group is called the ordinary pair (S, $(\cdot)_\Gamma$) where S is a non empty set and $(\cdot)_\Gamma$ Γ - multiplication in S such that

$$\forall (a,b,c,\alpha,\beta) \in S^3 x \Gamma^2, (a\alpha b)\beta c = a\alpha(b\beta c)$$

If in S' for a fixed α of Γ we define the operation (α) by the equivalence a(α)b = aαb, then (S, (α)) is a semigroup which is shortly denoted S_α. At [6] is proved this proposition:

If S_α is a group for a α∈Γ, then S_α is a group for every α ∈ Γ.

In case of Γ -near – ring it is not true the proposition analog to the proposition that we just mentioned. In other words, generally if for one α near-ring (M, +,(α)) is a

near-field then it does not derive that for every β∈Γ, near-ring (M, +, (β)) is a near-field. The point that we just made is shown in this simple counterexample.

Counterexample [7]. If we take Γ = Q and in the group of sum of rational numbers (Q, +) we define the multiplication with elements in the middle again rational numbers $(\cdot)_Q$ through the equivalence a γ b therefore (Q, +, $(\cdot)_Q$) is a Γ-near–ring. For a α ≠0 near-ring Q_α = (Q, +,(α)) is near-field, whereas for α = 0, Q_0 = (Q, +, (0)) is not a near-field.

Hence, in the analogy with Γ-semigroups the natural definition of near-field would be:

Definition 3.1.[7] A Γ-near-ring is called Γ-near -field if for every α ∈ Γ, the near-ring

M_α = (M, +,(α)) is near-field.

It is clear that when a Γ-near-ring is α-near– field according to [2] therefore it is Γ-near-field according to the definition 3.1. Conversely, generally it is not true, as it is shown in this counterexample:

Counterexample 2. [7]. Let (Q, +) be a sum group of rational numbers and Γ = Q* the set of rational numbers different from zero. The set Q forms a Γ-near-ring in relation with the general sum of the rational numbers and Γ-multiplication $(\cdot)_\Gamma$ if a γ b is nothing but a usual production of i a, b ∈ Q and $\gamma \in$ Q*.

Γ-near-ring ((M, +, $(\cdot)_{Q^*}$) is Γ -near- field according to **definition 3.1**, because for every α ∈ Q* near-ring

Q_α = (Q, +,(α)) is a field, consequently a near-field.

In fact, the ring (Q, +,(α)) is commutative, different from the zero ring and every element a ≠0 of Q has a for a inverse element the rational number $\frac{1}{\alpha a}$. But Γ-near ring (Q, +, $(\cdot)_{Q^*}$) is not a Γ-near-field according to [1].

In fact, if this Γ-near-ring would be a Γ-near-field according to definition in [1], therefore it would have a unique identity element **e**. Hence, for every 0 ≠ α ∈ Q we would have α·α·e = α, that is to say e = $\frac{1}{\alpha}$ for every α∈Q*, something that is in contradiction because we would have e = $\frac{1}{1} = \frac{1}{2}$ (!)

Here are some results on gamma-near-rings throught these new defition.

Lemma 3.2. **Let (M, +, $(\cdot)_\Gamma$) be in a Γ-near-ring zero-symmetric that has more then one element.**

The following propositions are equivalent:

(i) **Γ-near-ring M is Γ-near– field according to definition 3.1.**

(ii) **For every α∈Γ exists a 0 ≠ d ∈ M that is α-distributive and for every m ∈ M* we have Mαm = M**

<u>Proof</u>. (i) ⇒ (ii). Given that M is Γ-near-field, for every α∈Γ, near-ring M_α=(M,+, (α)) is quasi-field. Hence, (M*,(α)) is a group, one of which we are denoting with **<u>e</u>**. The element **<u>e</u>** is different from zero, because if e was zero for every m ≠ 0 we would have:

$$m = m(\alpha)e = m\alpha e = m(\alpha)0 = 0$$

(!)

Element **<u>e</u>** is α-distributive since the equalities are true:

$$e\alpha(a + b) = e\ (\alpha)\ (a + b) = a + b = e(\alpha)a + e(\alpha)b = e\alpha a + e\alpha b$$

For every m ∈ M* we have M*(α)m = M* and consequently M*αm = M.

Thus, Mαm = M since mα0 = 0, because M is Γ-near-ring zero-symmetric.

Hence, is true (ii).

(ii) ⇒ (i). Let α be an element whatever of Γ. For every two elements a, b of M* exist elements a', b' of M* such that b'(α)b = b'αb = a and a'(α)a = a'αa = b'. Therefore:

$$a'\alpha(a\alpha b) = (a'\alpha a)\alpha b = b'\alpha b = a \neq 0$$

Hence we have:

$$a \neq 0 \wedge b \neq 0 \Rightarrow a\alpha b \neq 0 \qquad (1)$$

Let d be an element α-distributive. Therefore it exists an element e∈ M such that eαd=d. Now we have:

$$(d\alpha e - d)\alpha d = d\alpha e\alpha d - d\alpha d = d\alpha d - d\alpha d = 0$$

And consequently, due to (1), dαe – d = 0 or likewise dαe = d.

If m ∈ M*, then:

$$d\alpha\ (e\alpha m - m) = d\alpha e\alpha m - d\alpha m = d\alpha m - d\alpha m = 0$$

Thus, dα(eαm – m) = 0 and consequently, due to (1) we have eαm – m = 0. Hence, eαm= m or likewise e(α)m = n.

In a similar way is shown that m(α)e=e. Thus, e is one of near-ring M_α = (M, +,(α)) .

Lastly, for every m∈M* exists a m' such that m'αm = e, or likewise m'(α)m = e. Hence,

M_α = (M, +,(α)) is near-field and consequently, since α is an element whatever i, M is a Γ-near-field.

Theorem 3.3. **Let M be a Γ-near-ring zero-symmetric that has more then one element. The following propositions are equivalent:**

(i) M is a Γ-near-field

(ii) M is B- simple and for every α exists an element that is α-distributive and for every 0 ≠m∈M exists an m'∈M such that m'α m ≠0.

Proof. (i) ⇒ (ii). We suppose that (i) is true. Let B be a bi-ideal of M different from zero and b ≠ 0 a element of B. It is clear that MΓb ⊆ M.

On the other hand, since (M, +,(α)) is near-field it exists identity element **<u>e</u>** and element b'∈M i such that b'α b = bα b' = e

Now for every m∈M we have

m = mαe = mα (b'αb) = (mαb')αb ∈ MΓb.

Hence M = MΓb. In the same way it is proved that b ΓM = M. From both equalities showed before we have M = MΓM = (b ΓM) Γ (MΓb) ⊆ bΓMΓb ⊆ B. Hence

M = B. Thus M is B- simple. The one e of near-ring

(M, +,(α)) is α-distributive since

e (α) (a + b) = a + b = e(α)a + e(α)b. For every

m ∈ M* exists an m'∈M such that m'(α)m = e ≠ 0. Thus, the proposition (ii) is true.

(ii) ⇒ (i). If (ii) is true, therefore firstly it exists an element which is α-distributive for every α. In the other hand for every m∈M*, Mαm is bi-ideal of M since Mαm is a subgroup of group (M, +) and there are true all the insertions:

(Mαm) ΓMΓ (Mαm) ⊆ (MαmΓMΓM)αm ⊆ Mαm

Bi-ideal Mαm is different from zero because m'αm ≠ 0. Hence, since M is B-simple, Mαm = M. Now, due to **Lemma 3.2**, for every α, (M, +,(α)) is a near-field and consequently M is Γ-near-field.

REFERENCES

[1] Chelvam,T. T., Meanakumari, N., On Generalizet Gamma – Near – Fields,Bull. Malaysian Math.Sc.Soc. (Second Series) 25 (2002) , 23 – 29.

[2] Chelvam,T. T., Meanakumari, N., Bi – ideal of gamma – near – rings , SEAM Bull. Math., 27 , (2004) , pp.983 – 988.

[7] Domi,E.,Petro,P., Gamma-near-fields and their characterization by quasi-ideals. International Mathematical Forum , 5,2010 , no. 3,

[3] [3]. Clay, J. R., Nearrings, Genesis and Aplications , Oxford University Press,1992 Survey of American Math., Soc. 7, Providence, R.I. 1961.

[4] Pilz, G., Near-Rings. The Theory and Applications, New York, 1977.

[5] Satyanarayana, Bh., A Note on gamma-near-rings, Indian J. Math. (B. M. Prasud Birth Centenary Commemoration volume) 41 (1999) 427 – 433.

[6] Sen, M. K., Saha, N.K., On Gamma-semigroups, I, Bull. Cal. Math. Soc., 78(1986).