On generalization of an semi-inner product function

Silvana Liftaj
Department of Mathematics
Faculty of IT, University of "A. Moisiu"
Durrës, Albania
silvanaliftaj@yahoo.com

Eriola Sila
Department of Mathematics
FSHN, University of Tirana
Tirana, Albania
eriola_liftaj@yahoo.com

Abstract— **The concept of inner product lays an important role in functional analysis and in its applications. Starting from its axiomatic, many researches have made various modifications passing in its generalization, such as semi-inner product. In fact, in this paper we define an semi-inner product function by modifying linearity concerning two variables of inner product, gaing a norm function and consequently a locally convex vectorial space.**

Keywords— inner product, semi-inner product, semi norm.

I. PRELIMINARIES

Before the explain the main result of this paper, we introduce some common know concepts.

Definition 1. The real vectorial space is $(X, +, \cdot)$ which satisfies the following axioms:

1. $\forall x, y$ in X, $x + y = y + x$.
2. $\forall x, y, z$ in X, $(x + y) + z = x + (y + z)$.
3. $\exists\, 0 \in X, \forall x \in X, x + 0 = x$.
4. $\forall x \in X, \exists x' \in X, x + x' = 0$.
5. $\forall x \in X$, $1x = x$ (where $1 \in R$).
6. $\forall k, l \in R$, $\forall x \in X$, $k(lx) = (kl)x$.
7. $\forall x, y \in X$ and $\forall k \in R$, $k(x + y) = kx + ky$.
8. $\forall x \in X$ and $\forall k, l \in R$, $(k + l)x = kx + lx$.

Definition 2. The normed vectorial space is a $(X, \|\ \|)$, where $\|\ \|$ is map of X into $\mathbf{R}$, which satisfies the following axioms:

1. $\forall x \in X, \|x\| \geq 0 \ \wedge\ \|x\| = 0$ if and only if $x = 0$.
2. $\forall \lambda \in \mathbf{R}, \forall x \in X$, $\|\lambda x\| = |\lambda|\, \|x\|$.
3. $\forall x, y \in X$, $\|x + y\| \leq \|x\| + \|y\|$.

Definition 3. The pseudo-normed (or semi-normed) vectorial space is a $(X, \|\ \|)$, where $\|\ \|$ is map of X into $\mathbf{R}$, which satisfies the following axioms:

1. $\forall x \in X, \|x\| \geq 0 \ \wedge\ x = 0 \Rightarrow \|x\| = 0$.
2. $\forall \lambda \in \mathbf{R}, \forall x \in X$, $\|\lambda x\| = |\lambda|\, \|x\|$.
3. $\forall x, y \in X$, $\|x + y\| \leq \|x\| + \|y\|$.

Definition 4. Let X be a real vectorial space. A inner product in a real vectorial space, which satisfy following conditions $(\ ,\ ) : X \times X \to \mathbf{R}$

1) $(x, x) \geq 0 \wedge (x, x) = 0 \Leftrightarrow x = 0, \forall x \in X$.
2) $(x, y) = (y, x), \forall (x, y) \in X^2$.
3) $(\lambda x, y) = \lambda(x, y), \forall (\lambda, x, y) \in \mathrm{R} \times X^2$.
4) $(x + y, z) = (x, z) + (y, z), \forall (x, y, z) \in X^3$.

II. INTRODUCTION

A basic concept in vectorial spaces is semi-inner product, which is defined by:

Definition 5. Let X be a real vector space. A semi-inner product is a mapping □ , □$: X \times X \to \mathbf{R}$ that satisfies the following properties:

1) □x, x□ $\geq 0, \forall x \in X$.
2) □x, y□ = □y, x□, $\forall (x, y) \in X^2$.
3) □$\lambda x, y) = \lambda$□x, y□, $\forall (\lambda, x, y) \in \mathbf{R} \times X^2$.
4) □$x + y, z) =$ □x, z□ + □y, z□, $\forall (x, y, z) \in X^3$.

Numerous efforts have been made for generalization of this concept. Thus, [5] was define the semi-inner product in a vectorial space X by modifying the condition (1), by omitting the condition (2), by permitting the linearity only for the first variable, and by adding the Cauchy–Schwarz inequality. This, in a topological viewpoint, leads in generation of normed spaces also, by setting $\|x\| = \sqrt{[x, x]}$, where [,] is symbol of semi-inner product.

By modifying the last axiom of the semi-inner product of Lumer with one of Hölder's inequality, [6] further generalize the semi-inner product by defining $[\ ,\ ] : X \times X \to \mathbf{R}$ which satisfies the following properties:

1) $[x, x] > 0$ for $x \neq 0$,
2) $[\lambda x, y] = \lambda[x, y]$,
3) $[x + y, z] = [x, z] + [y, z]$,
4) $[x, y] \leq [x, x]^{\frac{1}{p}} \cdot [y, y]^{\frac{p-1}{p}}$, where $p \in]1, +\infty[$.

This function which is called semi-inner product of type (p) defines a norm function by setting $\|x\| = \sqrt{[x, x]}$. In 2010 [7] generalize further the semi-inner product function with condition $[x, y] \leq \varphi[x, x]\ \psi[y, y]$, where φ and ψ are two specific function of $\mathbf{R}^+$ to $\mathbf{R}^+$. If $\varphi(t) = \psi(t) = \sqrt{t}$ obtained semi-inner product function of [5]. Further, if $\varphi(t) = t^p$, $\psi(t) = t^{\frac{p-1}{p}}$ obtained semi-inner product function of type (p).

For φ and ψ that satisfies the condition $\varphi(t)\psi(t) = t$ we take concrete shapes of semi-inner product functions. In 1981 [4] studied H-locally convex spaces which obtained from Hilbertian semi-norms obtained from a collection of semi-

inner products satisfying the following inner production properties:
a) $(x, x) > 0$ for all $x \neq 0$,
b) $(x, y) = (y, x)$ for all $(x, y) \in X^2$,
c) $(\lambda x, y) = \lambda(x, y)$ for all $(\lambda, x, y) \in \mathrm{R}\times X^2$,
d) $(x + y, z) = (x, z) + (y, z)$ for all $(x, y, z) \in X^3$.

III. MAIN RESULTS

In this paper we will treat the generalization of inner product functions, called a-pseudo inner production, which define a norm in a real vectorial space.

Definition 6. The function $(\, , \,) : X \times X \to \mathbf{R}$ is called a-pseudo inner production if it satisfy the properties:
1) $(x, x) \geq 0$ for all $x \in X$ and $(x, x) = 0 \Leftrightarrow x = 0$,
2) $(x, y) = (y, x)$, $\forall\ (x, y) \in X^2$,
3) $|(\lambda x, y)| \leq |\lambda| \sqrt{(x,x)}\ \sqrt{(y,y)}$, $\forall\ (\lambda, x, y) \in \mathbf{R} \times X \times X$,
4) $|(x_1 + x_2, y)| \leq \left(\sqrt{(x_1,x_1)} + \sqrt{(x_2,x_2)}\right)\sqrt{(y,y)}$,
$\forall\ (x_1, x_2, y) \in X^3$.

We illustrate this new definition, given by us, with following example.

Example 1. Let (X, p) be a normed space. From Hanh-Banach Theorem for all $x \neq 0$ there is a continuous linear form $f_x : X \to \mathbf{R}$ such that $f_x(x) = p(x)$.

For $x = 0$, $f_{x=0} \equiv 0$, we have $|f_x| \leq 1$.

Note $(x, y) = f_x(y)\, f_y(x)$. We construct the function $(\, , \,) : X \times X \to \mathbf{R}$ defined by

$$(x, y) = f_x(y)\, f_y(x), \text{ for all } (x, y) \in X^2.$$

We have:

- $(x, x) = f_x(x)\, f_x(x) = f_x^2(x) = p^2(x) \geq 0$.
- $(x, y) = f_x(y)\, f_y(x) = f_y(x)\, f_x(y) = (y, x)$, for all $(x, y) \in X^2$.
- $|(\lambda x, y)| = |\, f_{\lambda x}(y)\, f_y(\lambda x)| = |\, f_{\lambda x}(y)|\ |f_y(\lambda x)| \leq \|\, f_{\lambda x}\|\ p(y)\ \|\ f_y\|\ p(\lambda x) \leq |\lambda|\ p(x)\ p(y) \leq |\lambda|\ \sqrt{(x,x)}\ \sqrt{(y,y)}$,
  because $(x, x) = p^2(x)$, hereof $p(x) = \sqrt{(x,x)}$.
- $|(x_1 + x_2, y)| = |\, f_{x_1+x_2}(y)\, f_y(x_1 + x_2)| = |\, f_{x_1+x_2}(y)|\ |\, f_y(x_1 + x_2)| \leq \|\, f_{x_1+x_2}\|\ p(y)\ \|\, f_y\|\ p(x_1 + x_2) \leq p(y)\ p(x_1 + x_2) \leq$
  $\leq [p(x_1) + p(x_2)]\ p(y) = \left[\sqrt{(x_1,x_1)} + \sqrt{(x_2,x_2)}\right]\sqrt{(y,y)}$.

Proposition 1. If $(\, , \,) : X \times X \to \mathbf{R}$ is a-pseudo inner product, then we can define a norm.

Proof. Let denote $q(x) = \sqrt{(x,x)}$ for all $x \in X$. The function $q : X \to \mathbf{R}^+$ satisfies the following properties:
- $q(x) \geq 0$ for all $x \in X$ and $q(x) = 0 \Leftrightarrow x = 0$ are clear.
- $q(\lambda x) = |\lambda|\ q(x)$, for all $(\lambda, x) \in \mathbf{R} \times X$.

Indeed, for all $(\lambda, x) \in \mathbf{R} \times X$ we have $(\lambda x, \lambda x) \leq |\lambda| \sqrt{(x,x)}\ \sqrt{(\lambda x, \lambda x)}$ from which we get
$\sqrt{(\lambda x, \lambda x)} \leq |\lambda|\ \sqrt{(x,x)}$, thus

$$q(\lambda x) \leq |\lambda|\ q(x) \tag{1}$$

On the other hand, for $\lambda = 0$, we have

$$q(0\, x) = q(0) = 0 = 0\ q(x).$$

For $\lambda \neq 0$, $q(x) = q\left(\lambda \cdot \frac{1}{\lambda} x\right) \leq \frac{1}{|\lambda|} q(\lambda x)$, then

$$|\lambda| q(x) \leq q(\lambda x), \tag{2}$$

From (1) and (2) we imply

$$q(\lambda x) = |\lambda| q(x), \text{ for all } (\lambda, x) \in \mathbf{R} \times X.$$

- $q(x_1 + x_2) \leq q(x_1) + q(x_2)$, for all $x_1, x_2 \in X$,

$(x_1 + x_2,\ x_1 + x_2) \leq \left[\sqrt{(x_1,x_1)} + \sqrt{(x_2,x_2)}\right]\sqrt{(x_1+x_2, x_1+x_2)}$

from where we get:

$$\sqrt{(x_1+x_2, x_1+x_2)} \leq \sqrt{(x_1,x_1)} + \sqrt{(x_2,x_2)},$$

$$q(x_1 + x_2) \leq q(x_1) + q(x_2),$$

So, the function q is a norm function in real vectorial space X.

REFERENCES

[1] M.A. Abo Hadi, Semi-inner product spaces of types (p), M. Sc. Thesis, Faculty of Sc. King Abdulazis Univ. (1983).

[2] J.R. Giles, Classes of semi-inner product spaces, Trans. Amer. Math. Soc., 129 (1967), 436 – 446.

[3] E. Kramar, Locally convex topological vector spaces with Hilbertian semi-norms, Rev. Roum. Math press et Appl., 26 (1981), 55 – 62.

[4] E. Kramar, Linear operators on H-locally convex spaces, ibid. 26 (1981), 63 – 77.

[5] G. Lumer, Semi-inner product spaces. Trans. American Math. Soc., 100 (1963), 29–43.

[6] B. Nath, On generalization semi-inner product spaces, Math. J. Okayama Univ., 15 (1971/72), 1 – 6.

[7] H. Zhang, Generalized semi-inner products and applications to regularized learning, University of Michigan, Ann Arbor, MI 48109, 2010.